A Perspective Into Perspective

Approaching Geometry
from an Artistic Viewpoint

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Introduction

> Recent Changes in the teaching of Elementary Geometry in England and America have made it more then ever necessary that those who are engaged in the training of the teachers should be able to tell them something of the growth of that science; of the hypothesis on which it is built; more especially of that hypotheses on which rests Euclid's theory of parallels; of the long discussion to which that theory was subjected; and of the final discovery of the logical possibility of the different Non-Euclidean Geometries.
>
> - H. S. Carslaw, 1955

In the typical high school curriculum, the concepts of non-Euclidean geometry are often neglected, or mentioned only as an afterthought. Historically, non-Euclidean geometry is a relative newcomer to the educational scene. As recently as the 20th century, the importance of teaching the concepts of non-Euclidean geometries has been recognized, as the quote above indicates.

The necessity of teaching non-Euclidean geometry at the high school level is one that I believe to be important. The standards set by the NCTM for grades 9-12 require that all students "analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships." I believe that art and the theory of perspective provides a natural connection to these topics. Furthermore, NCTM calls us to have our students "visualize three-dimensional objects and spaces from different perspectives and analyze their cross sections", and, indeed, "use geometric ideas to solve problems in, and gain insights into, other disciplines and other areas of interest such as art and architecture."

This project will provide a way to meet these standards in a way that combines the idea of a realistic representation for art with the study of geometry. In particular, we consider the study of projective geometry and the invariant properties under projection. This MST project will focus on the combination of art and geometry to show students the relationship between them. Let's start with the big idea, perspective.

What is perspective? Perspective in art concerns the problem of representing a 3-dimensional space on a 2-dimensional flat surface. The idea of perspective can be traced back to ancient times and many of the techniques developed over the centuries are still relevant today in modern art. Perspective is from the "viewpoint of one eye" and is mathematically based on an approximation of how the eye actually works. As such, the topic of perspective is historically related to the study of Optics. In viewing an object, light travels off any surface to the eye in a straight line. These rays define size, shape, color and tone. In relating to perspective, an artist can imagine their canvas as a glass window through which these rays approach the eye. The artist will attempt to represent accurately, on this 2-dimensional canvas, a 3-dimensional world. Originally, perspective was created naturally from a set of rules used by artists and was not mathematically derived. However, we now know that perspective can be mathematically justified and analyzed from a purely geometric point of view. We are going to look at the development of the ideas and accomplishments -- both mathematical and artistic -- that were achieved. To understand the connection between mathematics and perspective, we will take a trip through the history of perspective and geometry and see how they came together during the Renaissance.

This paper consists of two major parts:

1. a mathematical and historical investigation of perspective in art and geometry
2. a curriculum which draws upon some of the concepts found in the investigation, and uses them to motivate the study of geometry

Although the mathematical investigation touches upon a number of more advanced topics, the curriculum is intended to be accessible to a typical high school geometry class.

Acknowledgements

In the creation of this project, the help and patience of others needs to be acknowledged. The class, I should mention, that sparked my idea for combining the idea of mathematical perspective and art, which was taught by Dr. Blieler. Furthermore, my interest in non-Euclidean geometries started with a class taught by Dr. Latiolais. I appreciate them for challenging me to think in a different way.

Equally important to the development of this paper was my husband, Aaron Lopresti. Obviously, without his help with the kids, I would not have found the time to complete my MST. Also, without his artistic contributions to this project, it would not have been as interesting to my students. I must also recognize his employer, Marvel Comics, who graciously allowed me to use their characters in my curriculum.

I also wish to thank my advisor, Dr. John Caughman. His support, advice, encouragement, humor, and appreciation of the "coolness factor" were invaluable throughout this process. Thank you.

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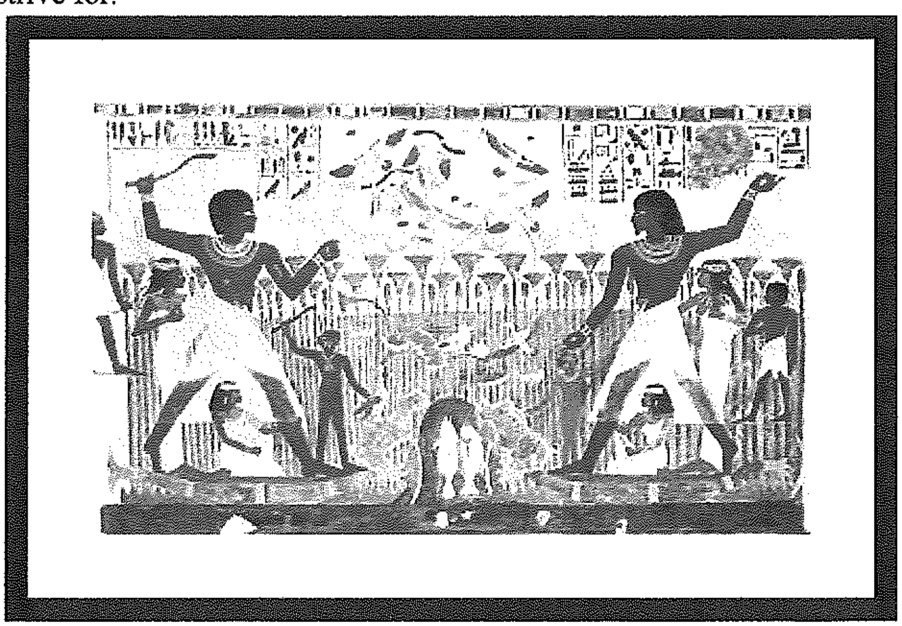
Part One:
Perspective in Art and Geometry

A History of Perspective in Art

To truly understand and appreciate the study of perspective, it is important to see how the ideas of perspective have changed throughout time. As we will see, each time period has its own political and religious ideas that determined the importance and respect for art and artists.

In ancient Egyptian time, the artist showed very little interest in having their art appear to be a realistic representation of space and depth. The idea of realistic representation would have been seen as antagonistic. Instead, Egyptian art represented strong religious and social symbols. Rather than showing perspective in their art, they often would only show "front panes" and depth was suggested by overlapping objects. This established when an object was in front of other objects; however, the relative size of the objects stayed the same. This meant that one could not tell how far in front or behind something was when compared to another object in the art. They also showed separate views within a single figure; for example they would show the profiles of the faces, arms, and legs, but the front view of the eyes and chest. In comparison to later works of art, as shown below, this was very primitive and lacked the realism that later artists would strive for.



The ancient Greeks began to use images that were based on how these items were actually viewed. This early form of perspective was based on observation and not mathematics. To show perspective, part of the figure or art was foreshortened but it was not complete. This meant that the part of the figure, which was closest to the "viewer", was larger in scale than the rest of the figure, and that objects farther away were much smaller.

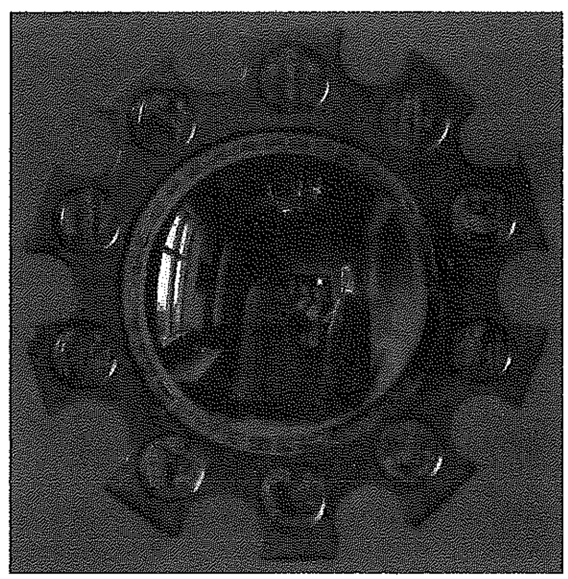
The Romans continued what the Greeks started, creating naturalistic and elaborate architectural perspectives. The Romans used an optical system for constructing three-dimensional space, with a fixed central point to which all lines "should naturally correspond, with due regard to the point of sight and the extension of the visual rays, so that by this deception a faithful representation of the buildings might be given…" (Cole, A., 1992, p. 10) During this time, theatrical scene painters developed a system that enabled them to realistically depict perspective based on the rules of optics. This was a novelty of this time period. Anaxagoras wrote about this scenery and the treatise that was written by an artist named Agatharchus. He wrote "in drawing the lines ought to be made to correspond, according to a natural proportion, to the figure which would be traced out of an imaginary intervening plane by a pencil of rays proceeding from the eye, as a fixed point of sight, to the several points of the object viewed". (Cole, R, 1976, p. 216) Although, the Romans clearly made great strides in understanding perspective, when their Roman Empire collapsed, the Byzantines reverted back to religious symbols and neglected all advances in art.

The next few centuries saw no significant advances in artistic representation. Indeed art from the Middle Ages appears to be flat when you look at it as the painting below illustrates.

St. Boris and St. Gleb, Byzantine art, artist unknown

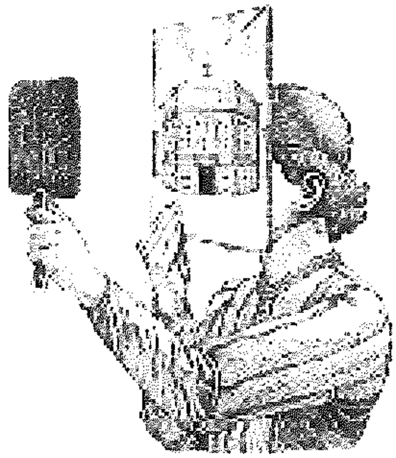
This effect was striking but not realistic to how we view the world. However, some modern artists use this today trying to achieve the same effect. The ending of the Middle Ages marked the beginning of what is now known as the Renaissance period. The dates of the beginning of the Renaissance were different for different areas but what were generally characterized the advances in art and sciences during this time.

During the 13th and specifically the 14th centuries, the idea of the canvas being like a window through which we observe our world started to re-emerge. This idea was sparked the artists' minds to accurately depict the world that they viewed and gave the artist a new credibility based on the technical aspect of their work. Italian artists began to combine the natural observation with simple measurement. During the same period, northern European artists solved many problems encountered by the Italians by studying nature, color and light. This was called the "method of judging by eye" (Cole, A, 1992, p. 10). Some northern European artists were known to use convex mirrors as a compositional aid, and some even copied images from the mirror as a way to create the effects of light, space and depth. An example of this approach is the *Arnolfini Marriage* as shown below.

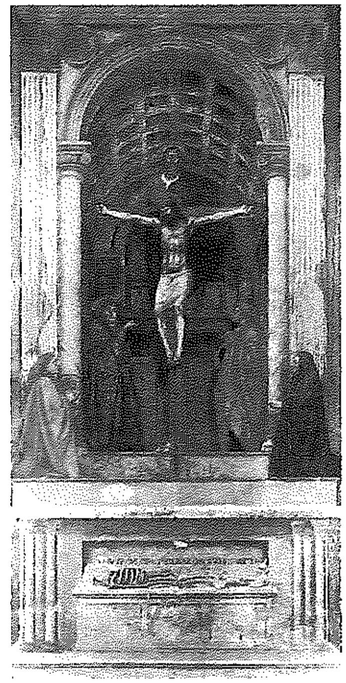


In this painting, depth is created by a sloping floor and beamed ceiling. The convex mirror used in this painting was actually included in the piece with the artist's reflection as well. The artists of this time were interested in the art of ancient Rome and the idea of a visual representation of the world around them. The creation of many of the ideas and rules of perspective developed during this time provided logic and consistency to the profession but still lacked a mathematical foundation. This marked the beginning of linear perspective.

The 15^{th} and 16^{th} centuries were a time of great invention and discovery that is now known as the Renaissance. By this time, an artist's skill was a precious commodity that was viewed as a noble profession instead of a medieval craft. Renaissance theorists explained perspective as the same principle as archery. The archer closes an eye to aim, just as the artists should imagine a straight line from his eye to the composition's vanishing point. Thus, perspective is "the theory of the one-eyed artist's vision". (Pedoe, 1976, p. 50) The idea that the eye perceives visual rays from an object, and that converge at the eye, came from the study of optics. This idea is known as the visual pyramid and dates back as far as the time of Euclid. Combining the ideas of optics and the construction of this visual pyramid was left dormant until the Renaissance. It was finally realized that the canvas could be placed in the pyramid, such that the rays intercepted the canvas, thus creating the picture. The credit of discovery or invention of this idea goes to the Florentines. Brunelleschi, a sculptor and architect, was first to demonstrate the principles of perspective in his famous "peepshow" experiments. Boring a hole through the vanishing point of the art created this peepshow. The viewer looked through the hole from behind the art and into a mirror so that they could see the reflection. This created an illusion of depth with the key being the vanishing point. An example of this is shown below.

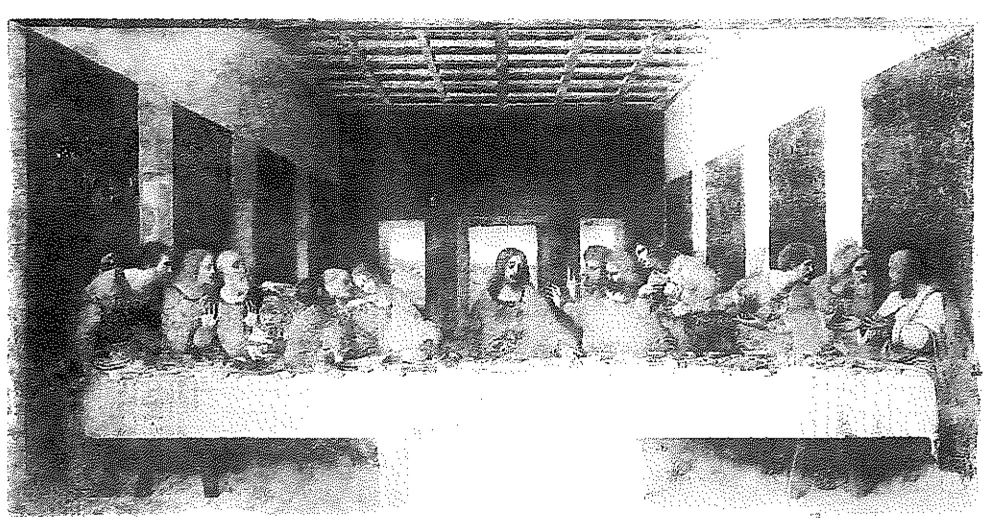


This experiment was of great importance to the work of a Florentine artist Masaccio (1401-28). In the church of Santa Maria Novella, Masaccio's *The Trinity*, shown to the right, is one of the earliest examples of rigorous linear perspective. It is believed that Brunellischi may have helped with its design.



Another artist, architect, antiquarian and man of letters was Leon Battista Alberti, who created methods for drawing in perspective. He developed a 3-stage technique for drawing in perspective. The details of this method are presented in the next chapter. Many believe that this technique may have been based on techniques used by ancient mapmakers. No study of perspective would be complete without looking at Leonardo da Vinci.

Leonardo da Vinci had a passion for scientific truths. He embraced mathematics, philosophy, architecture, engineering, sculpture, science, music, and painting. He is quoted as saying "Those who are enamored of practice without science are like sailors who board a ship without rudder and compass, never having any certainty as to whither they go" (Cole, A, 1992, p. 24) He studied "natural" perspective based on the way that space and distance are seen by the eye. He followed Alberti's method but also sought to improve it. He then created a new type of perspective; called "Curvilinear Perspective", based on the fact that human field of vision is curved. His painting of *The Last Supper* is full of ambiguities by the use of perspective and by the lack of it as well. To begin, Christ is painted in larger scale than the disciples and the vanishing point is the right eye of Christ. Also, the table is placed to cover up a strangely sloping floor but placed on top without addressing



the idea of space, as it is too wide. This painting is more of a suggestion of perspective rather than logical perspective as shown below.

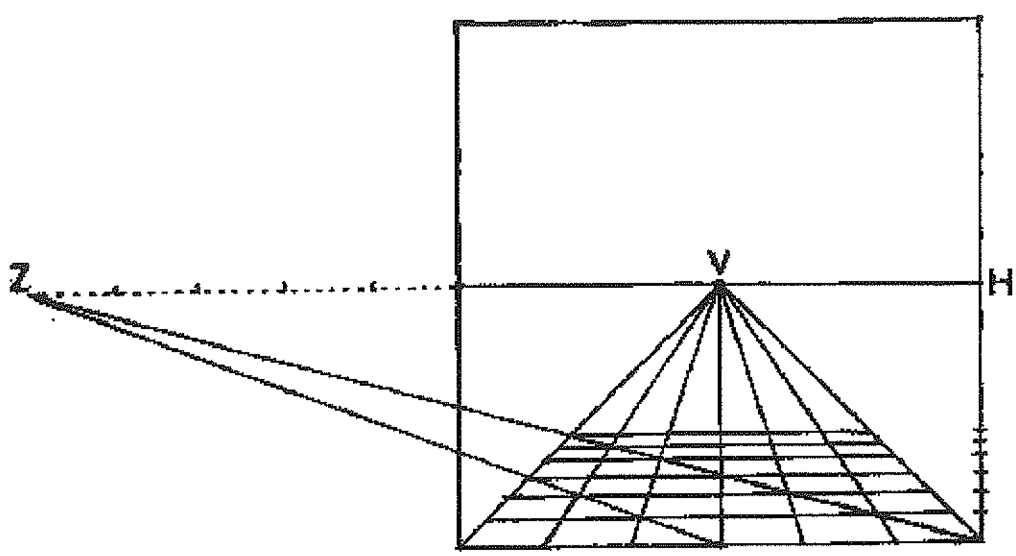
Lastly, Albrecht Durer, who loved art and geometry, did not come from a wealthy family or have an extensive education, but strived to fully understand the study of perspective. In 1506, he traveled to Bologna to learn the "secret" art. It was believed that the theoretical foundations of perspective had been known by a select few for over 50 years in Italy. Durer wanted to be part of that select few. He believed that geometry was the proper foundation of painting and strove to teach others in that way. During his life, he wrote several books on the study of geometry and was known as "the" natural geometer. He also developed tools to help artists create art in correct perspective. This perspective was known as mechanical and not mathematical perspective. The tools developed by Durer will be visited in the next chapter.

During the Romantic period of the late 18^{th} and early 19^{th} centuries there was a new emphasis on the artists' imaginations. The use of perspective was used to show infinity or landscapes of epic scale. This combined the idea of linear perspective with the aid of color and light. William Blake (1757-1827) was opposed to the "slavish mathematics" of perspective. He said that the rigorous training for perspective and life drawing "killed the imagination". Others like JMW Turner (1775-1851) believed that the rules were "turgid" but believed that perspective was useful for showing "amplitude, quantity and space". He chose to combine perspective with natural sciences. Still another was John Martin (1789-1854) who described his compositions as "perspectives of feeling". He used perspective and nature as well to create epic landscapes. One critic wrote, "Mr. Martin must have been born with prisms for eyes." His obituary said he "ravished the sense of the multitude and sometimes dazzled the imaginations of calmer men." (Cole, A, 1992, p. 46-47) During this era, some artists began to rebel against the idea of holding a mirror up to nature and would rather convey a feeling or make a statement with their art.

As we look at the modern eras, we see that some artists embrace the idea of perspective, as it has become a part of our vocabulary. It has been thoroughly used in mass media through movies, advertising, photography and even comic books. Artists such as Picasso and Beckmann have transformed the direction of modern art away from perspective. Their works create feeling by using images that contradict perspective. Artists such as Salvador Dali and MC Escher use perspective to create illusions that rely on our own expectations to confuse the brain. In modern times, both styles can be embraced on their own merit.

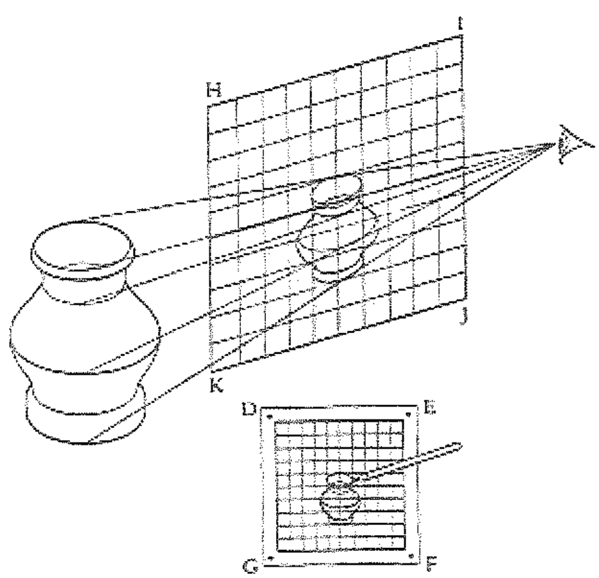
Tools of Perspective

As the study of perspective developed, artists created tools to help them create realistic images. To begin, we consider the 3-stage method from Alberti. Stage 1 of this method was based on the *braccio*, which was a unit of measure during the Renaissance. He started by drawing a rectangular picture area; he then divided the ground level into braccia. He then picked his vanishing point, which was 3 braccia from the center of the ground line, and drew a line from each braccia to the vanishing point that were called *orthogonals*. Stages 2 and 3 were more complicated. He next drew a side view including the picture and ground planes. He then divided the ground plane, on the right of the picture plane, into braccia and established the viewpoint to the left of the picture plane. He then connected points on the ground plane to the viewpoint. Where these lines crossed the picture plane determined the horizontal lines in the picture plane. These were then added to Stage 1, which created a "foreshortened chequered pavement". (Cole, A, 1992, p. 13) The horizon line is then drawn through the vanishing point parallel to the picture base. To check for accuracy, Alberti then drew a diagonal from the left bottom corner to the far right top corner. This was referred to as his "best method" in his treatise On Painting in 1435.

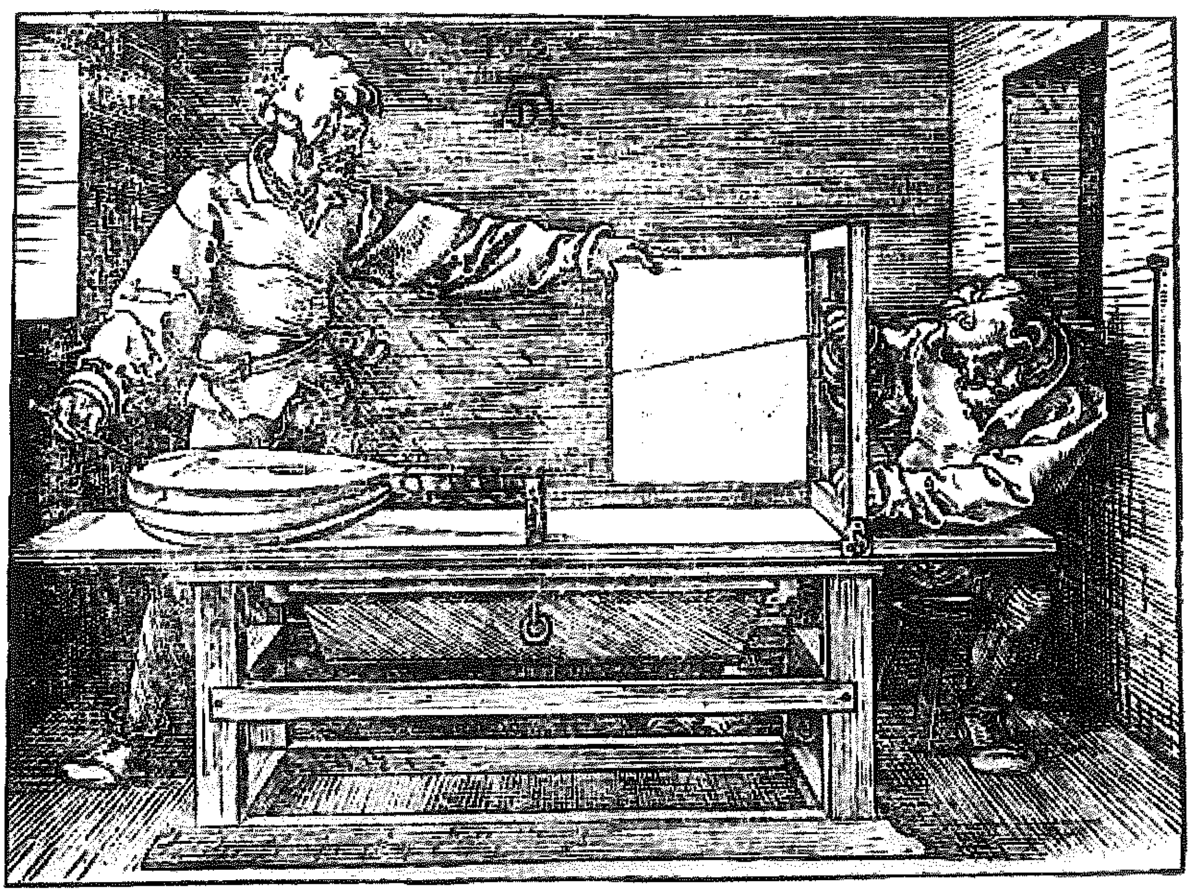


Along with the 3-Stage method, Alberti also created what is known as the Alberti veil. He made a "veil of loosely woven thread, dyed whatever color you please, divided up by thicker threads into as many parallel squares as you like, and stretched on a frame." He then "set this up between the eye and the object to be represented, so that the visual pyramid passes through the loose weave of the veil." (Kapler & Kapler, 2003, p. 202-203) This visual pyramid or cone was the key, which were actually two pyramids with the apex of one being the eye and the other the vanishing point with the veil in the middle. Along the same lines as the Alberti veil, Leonardo da Vinci used a device that consisted of a square of black thread stretched over a frame. An eyepiece is positioned from a distance twice the height of the frame. As the artist looks through the eyepiece, he then copies the outline of the form onto a drawing surface.

A very important method that Alberti used, based on the grid, was to find the correct shape for a circle in perspective. If you place a circle on a square grid, you can mark the intersection of the circle and the grid lines. After creating the grid with a perspective view, you can the reconstruct the circle by using the points of intersections. The circle is now an ellipse. At the time, the importance of this projection was lost and it would be a long time before the founders of projective geometry appreciated it.



Other drawing methods that have been previously mentioned are Brunelleschi's Peepshow and the use of convex mirrors. Another aid was created by Albrecht Durer (1471-1528) who believed in the importance of a mathematical foundation for creating perspective. When he wrote a treatise called "Course in the Art of Measurement with Compass and Ruler", he devoted an entire section to perspective. He also created a device for producing images in perspective. Instead of an eyepiece, as in the aid created by da Vinci, he put a needle in the wall to represent the eye. He then pulled the string though a frame and put the tip on the object at the point where the eye is focused. This string was kept taut by a weight attached to the string at the needle. This string passes thorough a frame that has two movable threads stretched across it. These threads are then moved to meet the string. The point of intersection corresponds to the point of focus on the picture plane. The string is removed and then a hinged shutter with paper attached is then moved into place over the threads. The point of intersection can now be marked on the picture plane. Obviously, this is done many times, but the viewpoint is fixed.



Artists were so eager to create devices to help them produce works with correct perspective that too many were created to mention. I have mentioned the ones that I found most interesting.

As we have traveled through time with an emphasis on art, we'll go back to the beginning of the history of geometry. Throughout time, ideas of nature and geometry and their connection have changed. However, the underlying question of whether the relationship between nature and geometry could be explained through mathematics has not.

A History of Perspective in Geometry

The roots of Geometry or earth measure were first sprouted in Babylon and Egypt. The idea of measure was first created in the sixth millennium B.C. in the Nile Valley where each year the river would flood the area and the people of the valley would evacuate to mounds that were like islands. When the river receded, the people would return for the dry season, which was the time for planting and harvest. By 3500 B.C., they had created industry and with industry and wealth inevitably come taxes. It is worth considering that taxes may have been the idea that generated the development of geometry. These taxes were based on the height of the flood and the surface area of each person's holdings. To survey the land for tax purposes, the *harpedonopta* or rope stretcher would have 3 slaves stretch a rope with knots at a specific distance. This would create triangles of specific sides and angles. The measures of these sides are what are referred today as Pythagorean triples. It seems that they used the Pythagorean theorem yet did not use it in an abstract form. In addition to using measurement to create taxes, they used them to create amazing structures. Of course, the most famous of accomplishments were the pyramids. It continues to amaze people that they were not only built but continue to exist today. To accomplish this, they developed an applied Egyptian geometry by necessity.

Another group that seemed to understand the Pythagorean idea were the Babylonians who created a math system that was considered more advanced than the Egyptians. Their mathematics was in written form without the use of math symbols that we use today in which they even had a list of the Pythagorean triples. Without an understanding of the idea of the theorem, it would have been impossible to generate this list. However, even though both groups showed the ability to use ideas of geometry, neither group cared to explore the relationships and more specifically how they work and why. That is until the Ancient Greeks came along and asked the tough questions.

The development of mathematics beyond calculations is credited to the philosopher Thales, who set the stage for mathematicians of future discoveries. Thales and others in India, China and Greece began a "revolution in human thought". (Mlodinow, 2001, p. 11) Rebelling against superstition and unorganized thinking, they began what is now known as the Golden Age. Thales traveled extensively to satisfy his thirst for knowledge. He traveled to Babylon and Egypt where he became somewhat a celebrity, and developed many techniques for measure, properties of triangles and even predicted the solar eclipse of 585 B.C. He began to organize a system for geometry to prove theorems and developed logical reasoning. He believed that we should be able to explain all in nature by mathematics. In his later years, he met a young man named Pythagoras. He apparently told Pythagoras to go to Egypt, which he did. In Egypt, he learned Egyptian geometry and Egyptian hieroglyphics. Later, he was in Babylon where he learned the Babylonian mathematics. When he returned home at the age of 50, he had brought both mathematics together and was ready to teach or really preach. The teachings of Pythagoras remained active until around 600 A.D., which started the time known as the Dark Ages.

Around the time 300B.C., a man named Euclid would create a work that would become one of the most read works of all time called *Elements*. His goal was to organize Greek understanding of geometry by using an innovative logical approach. First, he made the definitions short, precise and concise. He then used the definitions to create axioms or postulates so that nothing unstated could be used. Lastly, upon looking at the relationships, he created logical consequences that came from only the rules of logic using the axioms and theorems already proven. This created the idea of exactness and rigor in a proof. As a result, "It is intuitively obvious" was no longer acceptable. Euclid's elements remained "sacrosanct" for over 2000 years.

> Already in his own, ancient time, Euclid's *Elements* was a classic. Euclid not only defined the nature of mathematics, but his book played a central role as a model of logical thought in education and natural philosophy. It was a key work in the intellectual revival of the Middle Ages. It was one of the first books printed after the invention of the printing press in 1454, and from 1533 until the eighteenth century it was the only one of all the Greek works to exist as a printed text in the original language. Until the nineteenth century, every work of architecture, the composition in every drawing and every painting, every theory and every equation employed in science were all inherently Euclidean. *Elements* was not unworthy of its great stature. Euclid transformed out intuition of space into an abstract logical theory from which we could make deductions. Perhaps most of all, we must credit Euclid with attempting to shamelessly bare his assumptions, and never pretending that the theorems he proved were

anything more than logical deductions from his few unproven postulate. (Mlodinow, 2001, p. 96)

In Euclid's *Elements*, he begins with what he refers to as the 5 common notions, which are then followed by his 5 postulates. All theorems are then deduced from just 10 assumptions. Euclid's 5 postulates are as follows:

1. A straight line may be drawn from a point to any other point.
2. A finite straight line may be produced to any length.
3. A circle may be described with any center and any radius.
4. All right angles are equal.
5. If a straight line meet two other straight lines so as to make the two interior angles on one side less than two right angles, the other straight lines meet on that side of the first line.

A more familiar version of the 5^{th} postulate is: There exists exactly one parallel line through a point not on a given line. This 5^{th} postulate, also referred to as the parallel postulate, caused problems for most scholars who studied Euclid, as it was not as simply stated as the rest. Many felt it was added at the end, as Euclid did not use it to prove his first 28 postulates. During the Renaissance, the mathematician Renee Descartes sought to make Euclidean Geometry easier and less taxing. He "translated space into numbers, and more importantly, used his translation to phrase geometry in terms of algebra." (Mlodinow, 2001, p. 82) He turned the plane into a type of graph by drawing a horizontal and vertical axis. Ptolemy used this idea as early as the second century but only in maps or geographical terms. Descartes took this idea further by not just assigning coordinates (coordinate pair) but by how he used this information. It is believed that Fermat had the same idea about the same time but since he was notorious for not publishing his work, he did not get credit. Thus what we now use in analytical geometry is known as the Cartesian plane and not Fermatian plane. Over the centuries, many mathematicians have come close to discovering a new kind of space by trying to prove the 5^{th} postulate. However, they were hampered by their own belief that the parallel postulate was necessary and absolute. When they came upon a stumbling block in the proof, they would throw it out for fear of retribution and ridicule from other mathematicians. Many mathematicians have attempted to prove the parallel postulate only to use what they assumed or to replace it with yet another assumption. This proof became known as the "fourth famous problem of geometry," (Boyer, C., 1991, p. 242) behind trisecting an angle, doubling a cube and squaring the circle.

Proving the Fifth Postulate

Over time, mathematicians have made many attempts at proving Euclid's 5^{th} postulate using only the first four axioms. Many found the 5^{th} postulate "not sufficiently evident to be accepted without proof". (Bonolo, 1955, p. 2) Early attempts were made by men known as Posidonius (1^{st} Century, B.C.) who called parallels two equidistant coplanar straight lines and Geminus (1^{st} Century, B.C.) who noticed that a different definition than Posidonius could exist to really deduce a definition for parallel lines. He pointed out that the behavior of the hyperbola with respect to the asymptotes could be in fact parallel while not being equidistant.

Another attempt was made by Ptolemy (2^{nd} Century, A. D.). He assumed an alternate form of the parallel postulate and then used it to derive the original thus using a circular argument. Proclus Diadochus made the next known attempt a couple hundred years after Ptolemy. His error was assuming that the distance between two lines is constant based on his own experience and notion of parallel, which was equivalent to the postulate. Another attempt made by Al-Nirizi (9^{th} Century) and yet another by Nasir-Eddin (1201-1274). Lastly, Thabit ign Qurrah (9^{th} century) assumed that the lines were straight, thus repeating Ptolemy's mistake. Many years passed before these works were looked at again.

In 1663, John Wallis (1616-1703) did not use the concept of equidistant lines being parallel but of similarity. Wallis' specialty was cryptography and he wanted to replace the "distasteful" parallel postulate with something more obvious. His idea was based on similar triangles. Any triangle can be enlarged or made smaller to any size without changing the measurements of the angles. Reversing his reasoning leads to the idea of if a space exists which the parallel postulate does not hold, then no similar triangles exist. This idea was troubling as we are used to seeing triangles everywhere and it would distort the idea of a scale model. Could such a space exist?

Other mathematicians of great importance who were interested in proving the 5^{th} postulate include Gerolamo Saccheri (1667-1733) and Johann Heinrich Lambert (1728-1777). Saccheri attempted to prove the 5^{th} postulate using a quadrilateral with one right angle and cases in which the angles remaining would be acute or obtuse. Lambert's argument was also based on a quadrilateral but with 3 right angles and the cases involving the fourth. Both of these proofs or arguments ultimately led to contradictions. These arguments eventually led others on the path to discovering the impossibility of proving the 5^{th} postulate.

The French Geometers of the 18^{th} century also continued forward with this "fourth famous problem of geometry" such as D'Alembert (1717-1783), De Morgan and Lagrange. This continued into the 19^{th} Century with Carnot (1753-1823) and Laplace (1749-1827). Carnot believed the route of parallels with the principle of similarity. However, Laplace relied on Newton's Laws of Gravitation to make his point. In a note he writes,

> The attempts of geometers to prove Euclid's Postulate on Parallels have been up till now futile. However no one can doubt this postulate and the theorems, which Euclid deduced from it. Thus the notion of space includes a special property, self-evident, without which the properties of parallels cannot be rigorously established. The idea of a bounded region, e.g., the circles, contains nothing which depends on its absolute magnitude. But if we imagine its radius to diminish, we are brought without fail to the diminution in the same ratio of its circumference and the side of all the inscribe figures. The proportionality appears to me a more natural postulate than that of Euclid, and it is worth of note that is discovered afresh in the results of the theory of universal gravitation. (Bonolo, 1955, p. 61)

Another mathematician worth mention is Adrien Marie Legendre (1752-1833). Legendre attempted to make the 5^{th} postulate into a theorem, which was very similar to the arguments of Saccheri. He is also worth mention for his writing, in which he actually added nothing new to geometry of the day, but was presented in such a way that was easy to read and understand. This helped to create an increase of numbers of those who were interested in these new ideas. By the end of the 18^{th} century, they could have concluded that non-Euclidean spaces may exist. Instead, their finding led to strange properties that also led to contradictions so they assumed the space must be Euclidean. It wasn't until middle of the nineteenth century that these ideas become known.

These ideas are what have been known as Anti-Euclidean, Astral Geometry and finally as Non-Euclidean geometry. This new geometry questioned the absolute necessity of the parallel postulate. The problems with these new ideas were that many were not published which led some mathematicians to work on these ideas at the same time in isolation. One such mathematician was named Friedrich Gauss (1777-1855).

Gauss, Riemann and the Sphere

Friedrich Gauss was born on April 30th, 1777. At the age of 12, he began to question Euclid's *Elements*, specifically the 5th postulate. He didn't want to replace it or make it easier to use or make it "unnecessary" by proving it from the others. Instead, he questioned the validity of it. He called the parallel issue a "shameful" part of mathematics. He suspected that the uniqueness of parallels was "merely postulated" so the opposite could just readily be postulated. Was space actually curved?

During his life, many mathematicians would write him letters informing him of their findings and ideas. He would respond with how interesting he found it and that he had already done that. Only after his death, this was found to be true. In one letter about his attempts at proving the 5th postulate by assuming it false he wrote, "As for me, I have already made some progress in my work. However, the path I have chosen does not lead at all to the goal which we seek, and which you assure me you have reached. It seems rather to compel me to doubt the truth of geometry itself. It is true that I have come upon much which by most people would be held to constitute a proof: but in my eyes it proves as good as *nothing*." (Bonola, 1955, p. 65)

At the age of 15, he became the first mathematician to accept the idea that a "logically consistent geometry could exist in which Euclid's parallel postulate does not hold." (Mlodinow, 2001, P. 112) The discovery of curved spaces brought out many questions. Is our space curved or is it like Euclid's? With such questions, the whole idea of Euclidean geometry came tumbling down. This began a new era not just for math but also for physics. His work created a new space called hyperbolic space where instead of one unique line that is parallel to a given line through a given point not on the line, as in Euclidean, there are many such lines. Also, he showed that in hyperbolic space, the angle sum of a triangle measures less than 180 degrees. The problem was that no one knew how to create a model of this space. This was later solved by Eugenio Beltrami and also by Henri Poincare. The model of this geometry created by Poincare used what is now known as the Poincare disc. This was not only a model for hyperbolic space; it was hyperbolic space in 2 dimensions. Later, it was discovered that other mathematicians, namely Johann Bolyai and Nikolay Ivanovich, created similar findings.

Around the year of 1816, Gauss was working in Germany trying to measure and map distances and landmarks. This was difficult for several reasons. Most importantly, the surveying equipment of 1816 had a limited range, therefore, the measurements had to be done in shorter segments. Each of these segments created a degree of error, which quickly added up to a large amount. It all came down to the fact that the surface of the earth is not a flat Euclidean plane. It would be like trying to wrap flat paper around a sphere. The mathematics that was born from this problem is now known as differential geometry. Gauss recognized two important features of this. One being that "a surface could be considered a space in itself" (Mlodinow, 2001, p. 128) and the other that the "curvature of a given space could be studied solely on the surface itself, without reference to a larger space that may or may not contain it." (Mlodinow, 2001, p. 128)

Another consequence of this was the creation of yet another space where another option was considered. If not one or many parallel lines then what about no parallel lines? This created what is known as elliptical space or spherical geometry. This had been studied to some degree by the Greeks and even by Gauss but they all failed to see the significance of an elliptical space. However, it had been proven that in Euclid's system that elliptical spaces cannot exist. Was it a problem with elliptical space or Euclid's axiomatic system itself?

The idea of elliptical or spherical geometry can be relative today to the model of the surface of a sphere or the earth's surface. This geometry does look like Euclidean geometry when looking at small distances or areas, however, in this geometry there are several interesting non-Euclidean facts. To begin with, it is imperative to define what we are beginning with. Point in fact; let's define a "point", "line" and "plane". The concept of a point is the same as in Euclidean geometry, but the term line in spherical geometry refers to geodesics or "great circles". Geodesics were known in ancient times, as they were used in ancient map making. A great circle is the largest circle where you could pull a string tight on a sphere and it will not move. Another way to think about it is to consider the equator or longitudinal lines of a globe as a great circle. These great circles would cut a sphere in exactly half through the center of the sphere. The term "plane" in spherical geometry refers to the surface of the sphere. Also in spherical geometry, it was discovered that the sum of the angles of a triangle is actually larger than 180 degrees.

The ideas of spherical geometry and that the globe is an elliptical space was discovered by Georg Friedrich Bernhard Riemann (1826-1866), who was a student of Gauss.

Riemann's first love was always mathematics so at the age of 19, while he studied theology at the University of Gottingen where Gauss just happened to be a professor, he changed to the study of math. At the age of 27, Riemann was working to become a lecturer at the university where the final step was to present a trial lecture. He was allowed to submit three ideas but he was only fully prepared for the first two. As luck would have it, Gauss chose his third topic. After a near breakdown, Riemann spent the next seven weeks preparing his lecture. On June 10th, 1854 he presented his lecture on how "the sphere could be interpreted as a two-dimensional elliptic space." (Mlodinow, 2001, p. 138) With hyperbolic and elliptical geometries in place, non-Euclidean geometries began the long road towards acceptance in the mathematical world.

From the Sphere to Projection: Girard Desargues

The ideas of elliptical and spherical geometry also led to the creation of projective geometry. The work was based on a thought derived from perspective in Renaissance art and from Kepler's principle of continuity. Kepler's principle focuses on the idea that a parabola has two foci with one at infinity. This idea of a point at infinity became a focus of projective geometry. For example, when a circle is viewed obliquely, it looks like an ellipse. Similarly, the shadow of a lampshade will be a circle or a hyperbola, depending on the angle of projection to the wall. "Shapes and sizes change according to the plane of incidence that cuts the cone of visual rays or of light rays; but certain properties remain the same throughout such changes, and it is these properties that Desargues studied. For one thing, a conic section remains a conic section no matter how many times it undergoes a projection." (Boyer, C, 1991, p. 359) However, this view was dependent upon the acceptance that the parabola had a focus at infinity and that parallel lines meet at a point at infinity. Under these projections, the properties which remained the same were of great interest. One person who brought together the ideas of Renaissance perspective with the study of these invariant properties was Girard Desargues who is the founding father of projective geometry.

Who was Girard Desargues? Unfortunately, there is not a lot of information available about the personal life of Girard Desargues. However, we do know that he was born in Lyon, France in 1591 to an affluent family. This afforded him many opportunities for education and leisure time to pursue his own interests like architecture and military engineering. He is credited with the invention of a new non-Euclidean geometry called *projective geometry*. This was his most important work that he wrote under the title, "Rough Draft of an Attempt to Deal with the Outcome of a Cone with a Plane". (Boyer, C., 1991, p. 359) A very small number of copies were printed in Paris in 1693, which were probably just for his close friends. The text remained undiscovered until a handwritten copy by Phillipe de Lahire was found in a Paris library. Unfortunately, it was hard to understand due in part to the new vocabulary. It was presumably written, not for math scholars, but for "mechanics and practical mathematicians." (Boyer, C. 1991, p. 361) It is believed that his study began when he was conducting business in Paris where he became part of a mathematics group. This group headed by Marin Mersenne and also included Rene Descartes and Blaise and Etienne Pascal. He may have written his math work for only those in the group, as he did not mass-publish his work. His unorthodox views of perspective and its role in architecture and geometry met with little favor. Therefore, he was left to figure it out on his own.

As mentioned above, in 1636, he wrote about perspective in what is known as the Brouillon Project or the Rough Draft. In his study of perspective, Desargues is credited with his own theorem obviously named the Theorem of Desargues. This theorem states: "If two triangles are so situated that lines joining pairs of corresponding vertices are concurrent, then the points of intersection of pairs of corresponding sides are collinear, and conversely." (Boyer, C., 1991, p. 361) Abraham Bosse, who was a friend of Desargues, first published this theorem in 1648. By the 19th century, this theorem became one of the "fundamental propositions of projective geometry" (Boyer, C., 1991, p. 361) We will consider this theorem more closely in a following chapter.

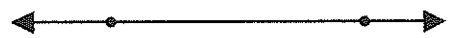
So it was not until the early years of the nineteenth century that Desargues' ideas were revisited and a formal geometry was developed which added as many vanishing points to the Euclidean plane as there are directions, and a circumscribing horizon as well, a "line at infinity" for all those "points at infinity" to lie on. This is the projective plane.

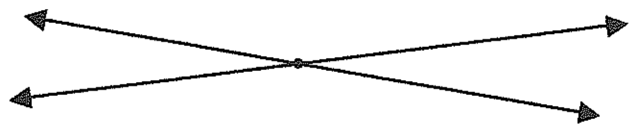
Projective Geometry

To completely understand what projective geometry is, let's go back to the beginning. What are the rules of a logical system? When creating a system you start with certain elements and try to deduce properties from them. Things that are accepted without formal definition are called undefined elements. The relationships between these elements that are without formal proof are called axioms or postulates, which can be thought of as the rules of the game. Following from these axioms, there are relationships that can either be proven or deduced, which are the theorems. Finally, we have definitions that relate to all the above-mentioned terms and relationships. The postulates of any theory or system must satisfy certain conditions before being generally accepted. It is important that they be easy to understand, few in number and use only a few accepted undefined terms. It is important that they also be *complete*, in the sense that you would like to have as many true statements as possible be provable from the axioms. A theory is considered *consistent* if it is possible to find a single concrete model that satisfied all of the postulates. What people have failed to see is that these sciences may not, at that moment, have a practical application but could in the future. I guess it goes back to the standard question in every classroom, "When are we ever going to use this?" Indeed, you never know.

Projective geometry is simply a geometry of collinearity and concurrence. In studying projective geometry, we focus on the invariant properties under projection. At the core of this geometry is projection and perspective. Therefore, the ideas of distance, angle measure and parallel lines are abandoned. To start defining projective geometry, we should begin with our undefined terms. Similar to Euclid, we have points, planes, and lines. If all these elements are on the same plane, it is called a projective plane.

Next, we look at the defined terms of projective geometry. If three points lie on the same line, they are collinear. If the points are on the same plane, they are coplanar. Points lying on the same line are called a range and lines containing a certain point are called a pencil. Two ranges are related to each other by a perspectivity with center O if the lines formed by corresponding points on the ranges are concurrent, or meet, at O. Perspectivity is what I will focus on in terms of projective geometry. Projective geometry, like elliptical geometry, is one where no parallel lines exist. What we would normally consider to be parallel lines are said to meet at a point at infinity. This point at infinity would correspond to the vanishing point in art, whereas the line at infinity would correspond to the horizon line on which the vanishing point lies. As fate would have it, the projective plane turned out to be simpler than Euclid's, requiring only 4 axioms:

1. Given any two distinct points, there exists a unique line containing these points. 
2. Any two lines contain at least one common point.

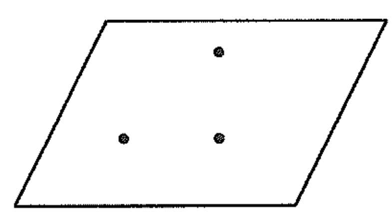


3. There exist three non-collinear points.

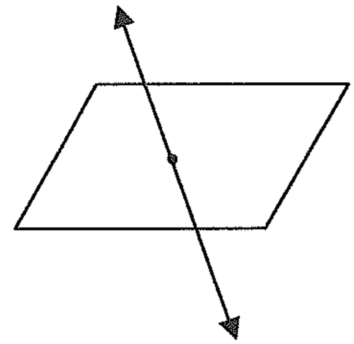
4. Every line contains at least three points.

If a set of points satisfies these axioms, it is by definition a projective plane. If we want to approach projective geometry from three dimensions, we would use the following six axioms to define a projective space:

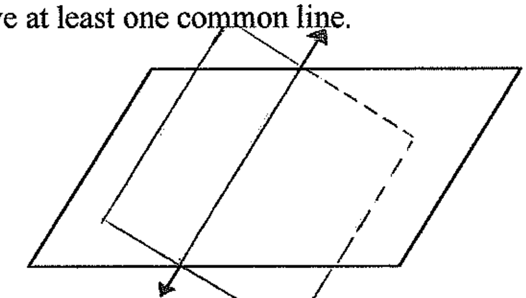
1. Given any two distinct points, there exists a unique line containing both these points. 
2. Any three non-collinear points lie on a unique plane.



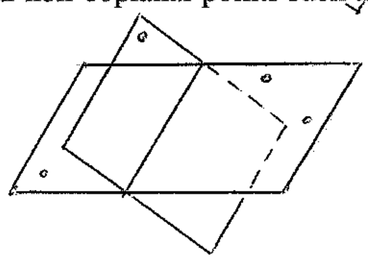
3. Any line intersects any plane in at least one point.



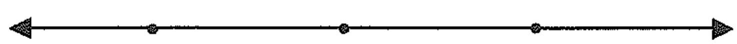
4. Any two planes have at least one common line.



5. There exist four non-coplanar points such that no three of them are collinear.



6. Every line contains at least three points.



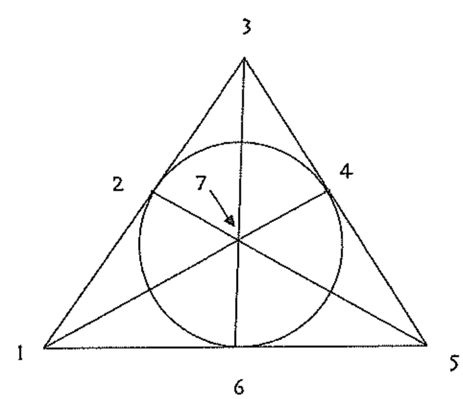
These axioms define a projective 3-space. It should also be mentioned that in a projective 3-space, every plane within it is guaranteed to be a projective 2-space.

A profound discovery in projective geometry is the principle of duality. The principle of duality on the projective plane states that the terms "point" and "line" are interchangeable as well as their appropriate verbs, "meet at" and "lie on", and the terms "point" and "plane" are interchangeable in projective space. This is a powerful tool in simplifying proofs in this geometry as it allows us to do half the work; which is truly a motivating factor for most mathematicians.

The principle of duality holds in projective geometry because all theorems and deduced from the axioms. Also, the dual of each axiom is either another axiom or follows immediately from the axioms. This means that any theorem about some relationship or configuration of lines and points will give us yet another theorem with the same structure about points and lines. (Notice the reverse of wording)

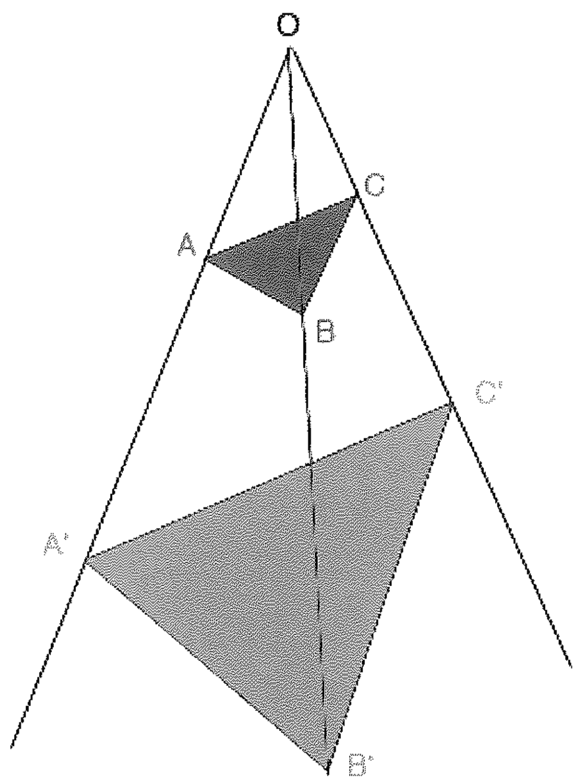
To prove the geometry is valid and consistent, you "only" have to create a model where all the axioms are met. Below are several models of the four axioms of a projective plane.

1. 7 points and 7 lines with 3 points on each (shown below).
2. 13 points and 13 lines with 4 points on each.
3. 21 points and 21 lines with 5 points on each.
4. 31 points and 31 lines with 6 points on each.
5. $n^2 - n + 1$ points and lines with n points on each for all $n > 3$ thus creating as many finite models of the projective plane as we want.



The Theorem and The Proof

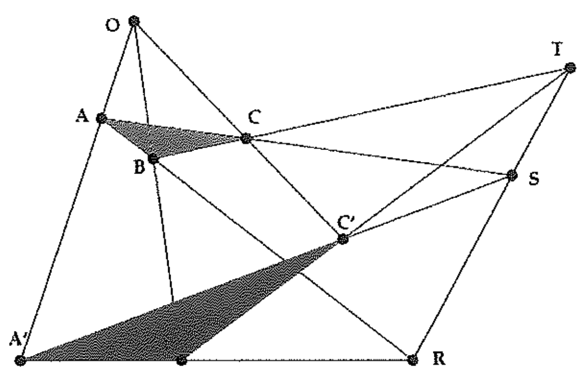
The theorem of Desargues is a fundamental proposition in the study of projective geometry. In the study of projective geometry, the core is perspective and projection. If we look at the most basic figure in geometry, the triangle and the relationship between two of them, as related to projective geometry, the Alberti visual pyramid gives us an immediate illustration.



If you look at triangle ABC from the point of view of point O, triangle A'B'C' is its image or shadow if O was a light. As well, if looking from the viewpoint of O, you would only see triangle ABC. In projective geometry, we say that the two triangles are in perspective when viewed from the center of perspectivity or O. To be in perspective from O, corresponding vertices must be collinear with O. So, OAA', OBB' and OCC' are collinear. We can now observe the concurrence of these lines at O. The problem that this created for mathematicians was that they could not say that the triangles are congruent or even similar as one triangle may be acute while the other obtuse. This was a problem as mathematicians of this time were used to congruence in geometry and equality in algebra. These are invariant properties of geometry and algebra that were readily accepted by mathematicians. This idea of invariance as applied to projective geometry brings us to perspective and projection. The question arises, "Is there any invariant remotely interesting or profound in this geometry?" In other words, this new geometry is "nice", but what can it do and how can we use it? This leads us to the theorem of Desargues. Desargues discovered a "simple, defining" situation in projective geometry and as well as a profound invariant. The theorem of Deseargues is stated as follows:

"If two triangles are so situated that lines joining pairs of corresponding vertices are concurrent, then the points of intersection of pairs of corresponding sides are collinear, and conversely." (Boyer, C., 1991, p. 361)

If we take the Alberti visual pyramid a step further by extending the sides of the triangles, we can create an illustration of the theorem. Notice that when extended, the corresponding sides AB and A'B' will meet at point R. In continuing this process, points S and T are created. This theorem states that points R, S and T will be collinear. We know that the extended sides of the triangles will meet as there are no parallel lines in projective geometry.

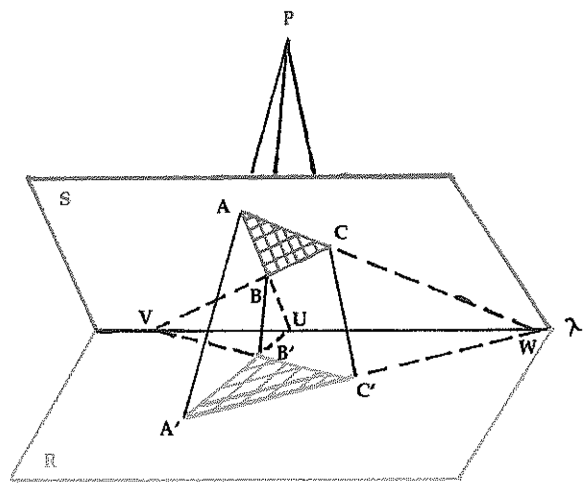


Curiously, the theorem of Desargues is not a theorem of the projective plane. That is to say, it cannot be proven from the four axioms of projective plane. Many geometers take the theorem as a fifth axiom of the projective plane. It is interesting to note however, that the theorem of Desargues CAN be deduced from the six axioms of a projective 3-space. It must be mentioned that once proven in 3-space, it can be assumed that the theorem will hold in a projective plane within that projective 3-space. In this section, we provide the proof of the theorem of Desargues from the axioms of the projective 3-space.

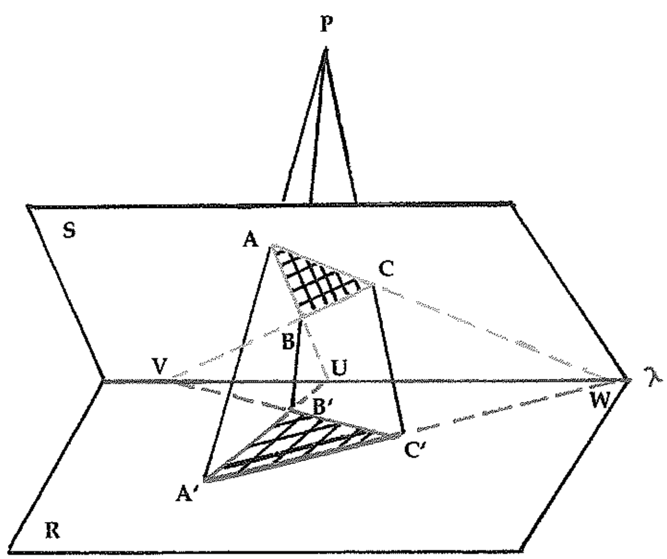
Here we go!

Proof: The proof in the projective 3-space will be done in two cases. The first case assumes the triangles are on different planes and the second case assumes the triangles lie on the same plane.

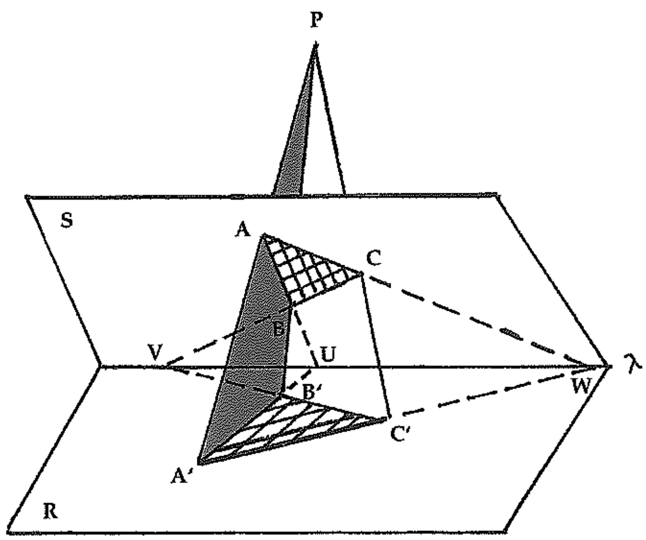
Case 1:



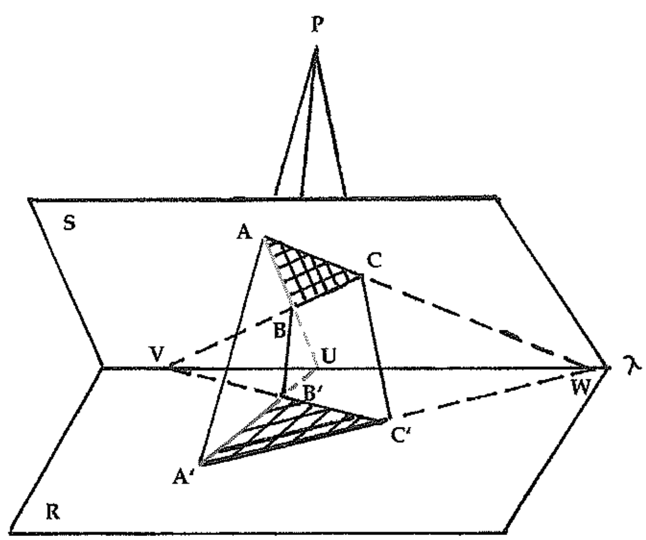
Case 1: Suppose two triangles ABC and A'B'C' lie on different planes S and R and are perspective from a point P which is on neither plane. We must prove that when the paired sides of the triangles are extended, they must meet at three points that lie on a line λ.



We claim the line of intersection of planes S and R is the desired line λ. In this Desarguean configuration, notice that the lines PAA' and PBB' intersect at P, thus creating a plane T.

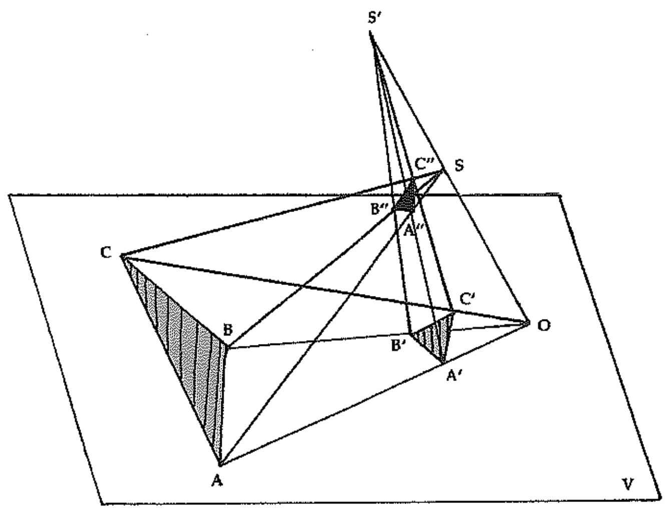


Next, AB and A'B' are two lines on plane T, so they must intersect at a point U.

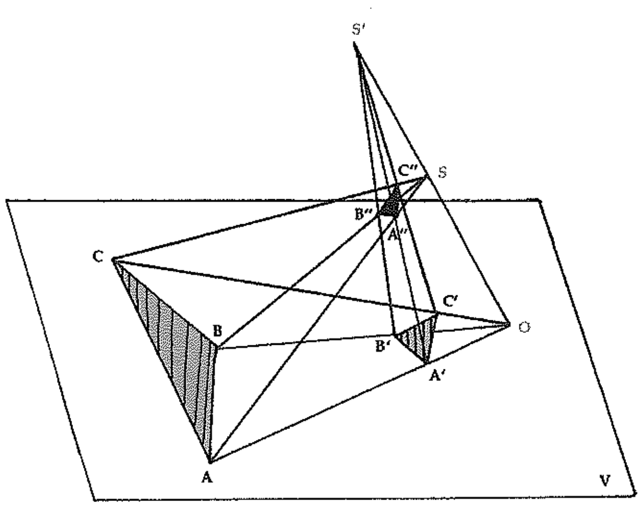


Since AB is on S and A'B' is on R, U is on both of these planes S and R so it must be contained in their intersection, which is line λ. Similar arguments can be made to show that points V and W, as shown in the diagram, also lie on line λ where the two planes intersect. Proving all three points lie on the line λ, as desired.

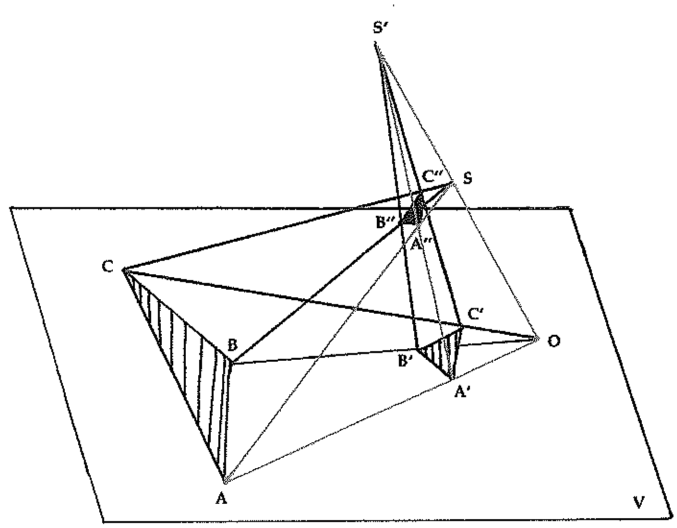
Case 2: Now we want to look at the results when both triangles are on the same plane. It is important to remember that any pair of planes within projective 3-space will intersect in a line. As a result, lines on these planes will also intersect at that line. In this case, we are given triangles ABC and A'B'C' on the same plane V that are perspective from a point O.



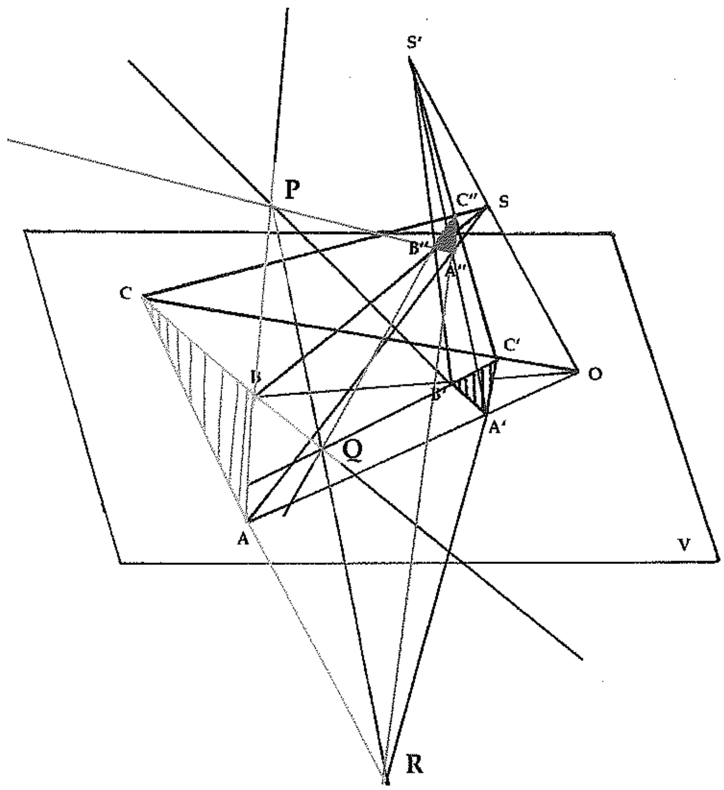
We know, from the axioms of projective space, that there exists a point S not on this plane. Now, there must be a line to S from O. Also, in projective geometry, every line must have at least 3 points so there is another point on this line, call it S'. Now, from S and S', we will build Alberti's visual pyramids to each of the triangles ABC and A'B'C' as indicated in the figure below.



So now we have our original configuration plus two Alberti pyramids. We next show that these pyramids intersect to create a triangle A"B"C".

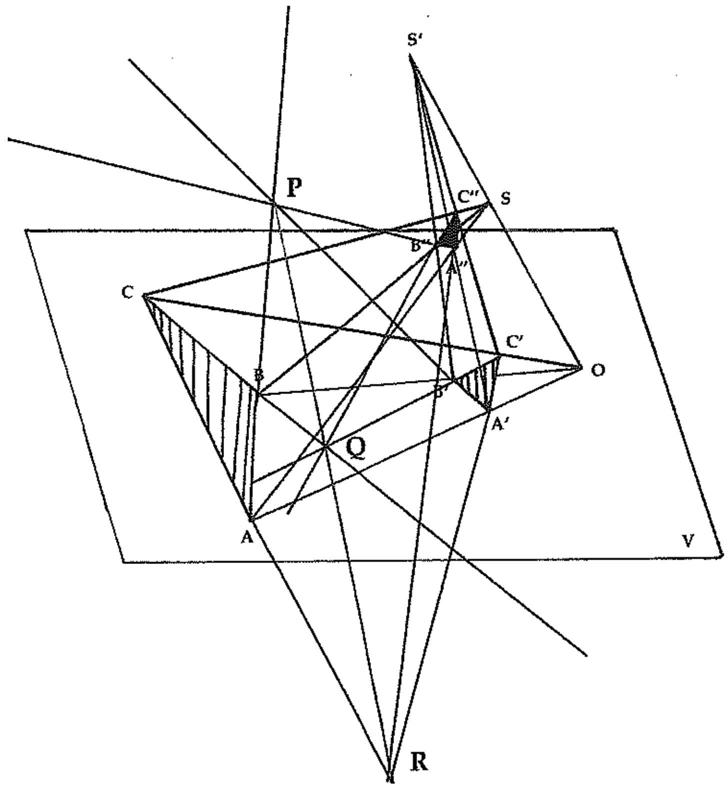


We know that these pyramids intersect to create triangle A"B"C". First, SA and S'A' must meet at a point A", due to the fact that OSS' and OAA' meet at O and create a plane. SA and S'A' are lines on this plane and must therefore intersect. Using the same argument, we can construct points B" and C". We join these points to form triangle A"B"C". As shown in the figure, triangle A"B"C" seems to be floating above plane V. Since ABC are not collinear and since ABC and A"B"C" are perspective from S, it follows A"B"C" are not collinear, thus creating triangles. Now, looking at A"B"C" and ABC, we observe that they are on different planes; as shown above in case 1, the corresponding sides, when extended, will meet at three points that are collinear. We'll call these points P, Q, and R as shown in the next illustration.



The line containing P, Q, and R will be called λ, and we note that it is the line of intersection between plane V and the plane created by A"B"C". Next, look at A"B"C" and A'B'C'. We note they are on different planes, so by a similar argument they too will create three collinear points. The line containing these points is again λ. In fact, these are actually the same three points. To see this, notice that since A"B" intersects λ at P and since A"B" intersects with A'B' on λ, they must be the same point, two lines must intersect in at most one point. Another way to think about it is that A"B"C" creates a plane that can intersect plane V at a line. Since triangles ABC and A'B'C' are on the same plane, then the "new" plane can only intersect plane V at one and only one line. This concludes case 2.

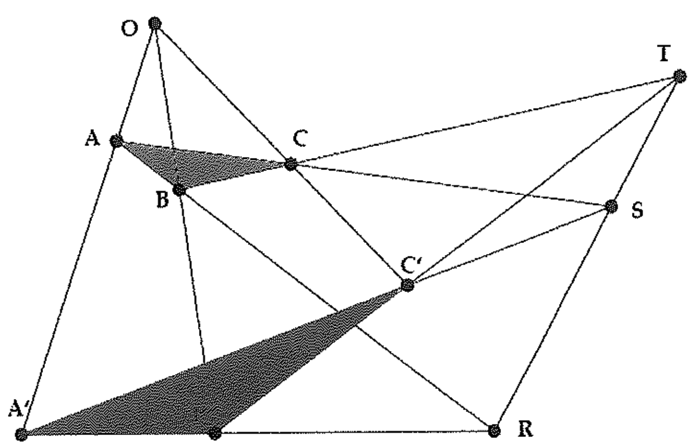
Thus, by cases 1 and 2 above, the Theorem of Desargues holds for any two triangles in a projective 3-space. QED



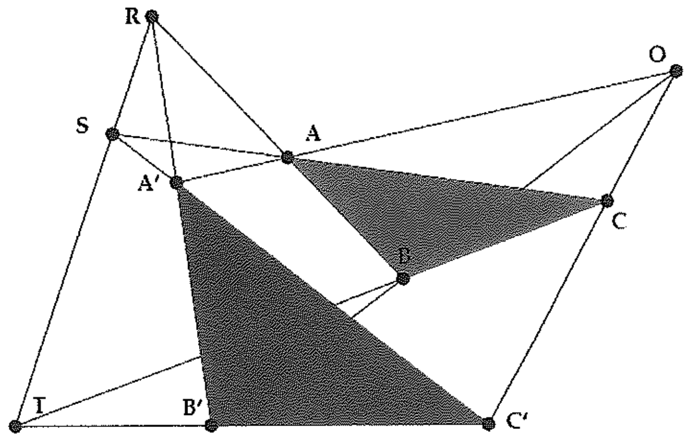
This proof was motivated by a proof in <u>The Art of the Infinite</u> by Robert and Ellen Kaplan.

An interesting note on the Desarguean configurations with the 10 points is that any of those 10 points can be the center of perspectivity O as shown below.

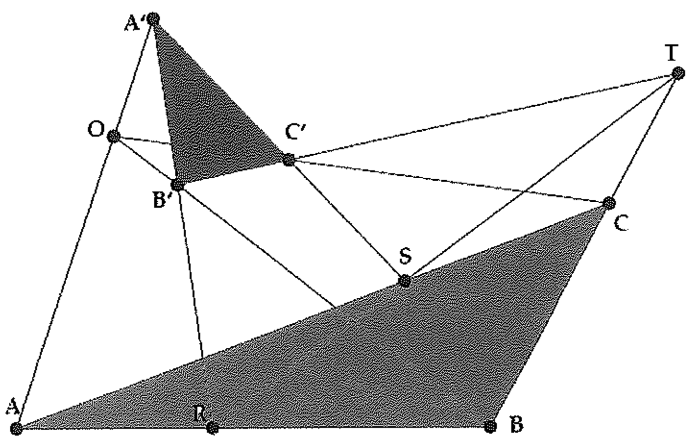
CASE 1:



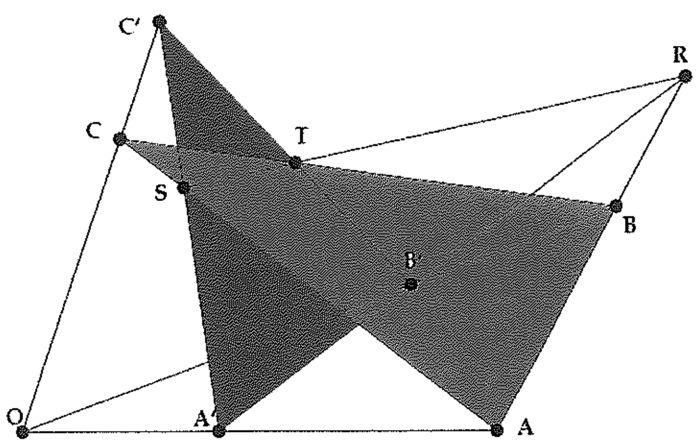
CASE 2:



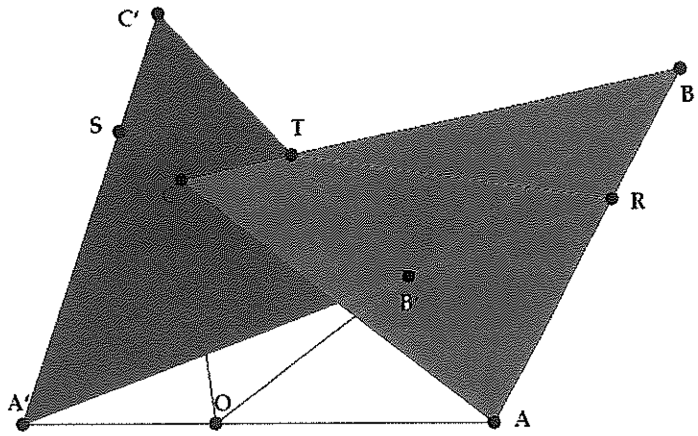
CASE 3:



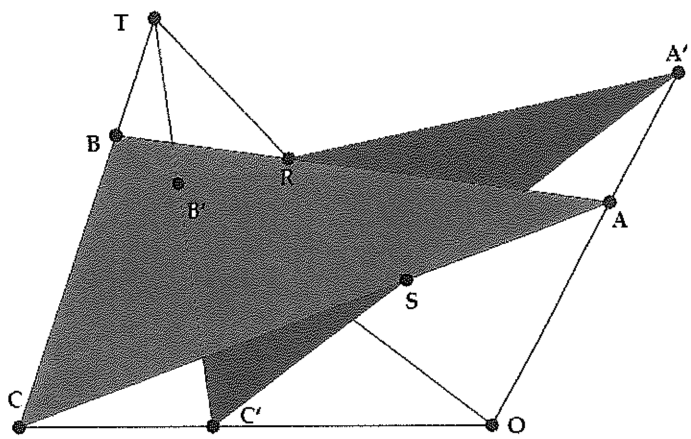
CASE 4:



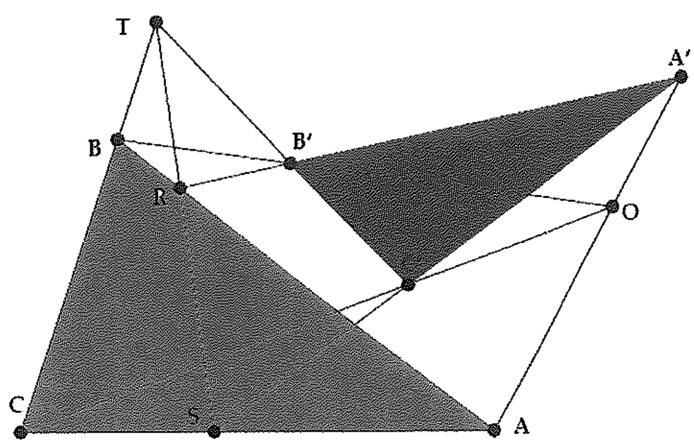
CASE 5:



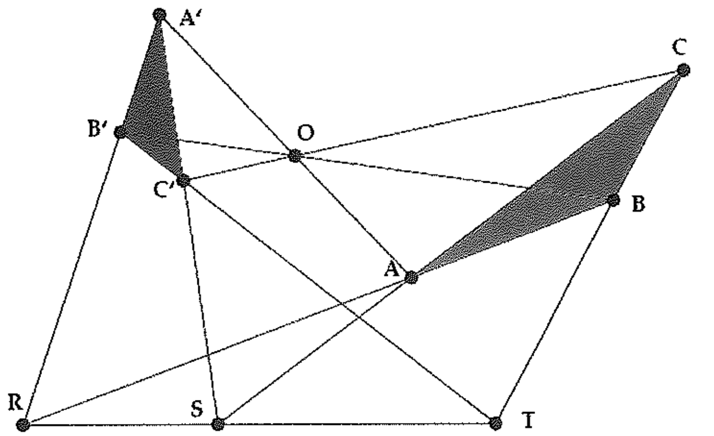
CASE 6:



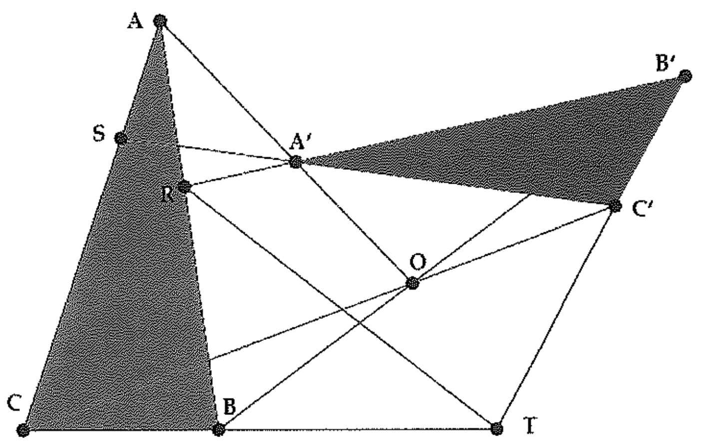
CASE 7:



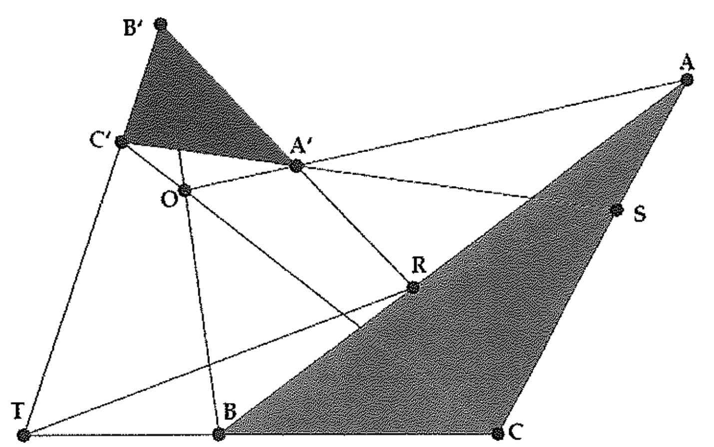
CASE 8:



CASE 9:



CASE 10:



Part Two:

Introducing Non-Euclidean Geometry Using Art and Perspective

Overview of the curriculum:

When starting this project, I knew that I wanted to explore a relationship between math and art. I decided that for my students, I wanted to help them to see beyond the Euclidean plane. Very little emphasis is placed on instruction beyond the Euclidean plane, which I wanted to address. To prepare my students for the new material, I have addressed non-Euclidean topics when appropriate throughout the school year. I had the privilege of teaching four section of geometry this year at Rex Putnam High School. Unlike my high school experience, I wanted my students to have the benefit of experiencing geometry beyond the typical curriculum.

When planning for instruction, it was important to create material that would be interesting to students but also help them to bring the two worlds together; art and mathematics. To begin, it was imperative that the students be given the historical content to appreciate how both the art and mathematics have changed over time. In order to accomplish this, I created a power point presentation and accompanying note guide in a format that was easily understood and organized. Each topic is clearly "boxed" with an accompanying comic book character.

The curriculum consists of 5 activities with a culminating project designed to give students a choice in how to convey the knowledge of perspective gained from this unit. This project gave students different choices of how to present their knowledge in the best way for them. Even though the mathematics behind this project is beyond the typical geometry curriculum, certain aspects of it lend themselves nicely to a high school student.

My main goal in developing curriculum for students is to introduce them to the idea of a non-Euclidean geometry. For all their years of math instruction, their ideas of line, point and plane are entrenched in the idea that a plane is flat, that there are parallel lines and that they go on forever without intersecting, and that non-parallel lines intersect at only 1 point. I wanted my students to make a connection between non-Euclidean geometries and how it is used in the real world namely in art.

During the school year, I have introduced ideas of non-Euclidean geometries when appropriate. In the beginning of the year, students were introduced to the origins of geometry and how Euclid's system is organized. Periodically, ideas from spherical geometry were introduced as well. When students were studying the Pythagorean theorem, I also introduced the concept of taxicab geometry. Finally, in this unit, the ideas were brought together to help students make connections.

Introduction to Activity 1

Previous Lessons:

September, 2005:

Introduction to Geometry: Lecture on basics of Euclid's organization of Euclidean geometry and how it is used with the Cartesian plane. I made sure that there was an emphasis on the basic components of a system. Specifically, that the system depends on what you define your basic components to be, what is considered to be a point, line or plane.

I am extremely glad that I chose to approach the beginning of the year in this manner. When presenting the material for this project, my students were already familiar with the concept of defining your basic terms. However, this did not transfer to their acceptance that not all lines are straight.

November, 2005:

When looking at triangle properties in the Euclidean plane, I chose to also introduce ideas from spherical geometry. I specifically addressed this with triangle sum. My students actually used constructions to show that the sum of the angles of a triangle (Euclidean) is 180 degrees. I then asked the question of how they think this might change in another geometry; specifically, the geometry on the surface of a sphere. My students readily accepted the idea of a geodesic but again struggled with the consequences of this now being our line. After looking at different drawings showing what it may look like, students started to see how this might work. Of course, no spherical geometry discussion would be complete without talking about air travel. When asking how this geometry is applicable to their life, using air travel made that question easy to answer.

This was an especially "passionate" day for the students. They realized how attached they are to the concepts of the Euclidean plane. It was great to see students debate with a passion for the topic. My only regret is that I could not spend more time on this topic and bring in the Lenart spheres to work with. That will be an activity that I hope to do in the future.

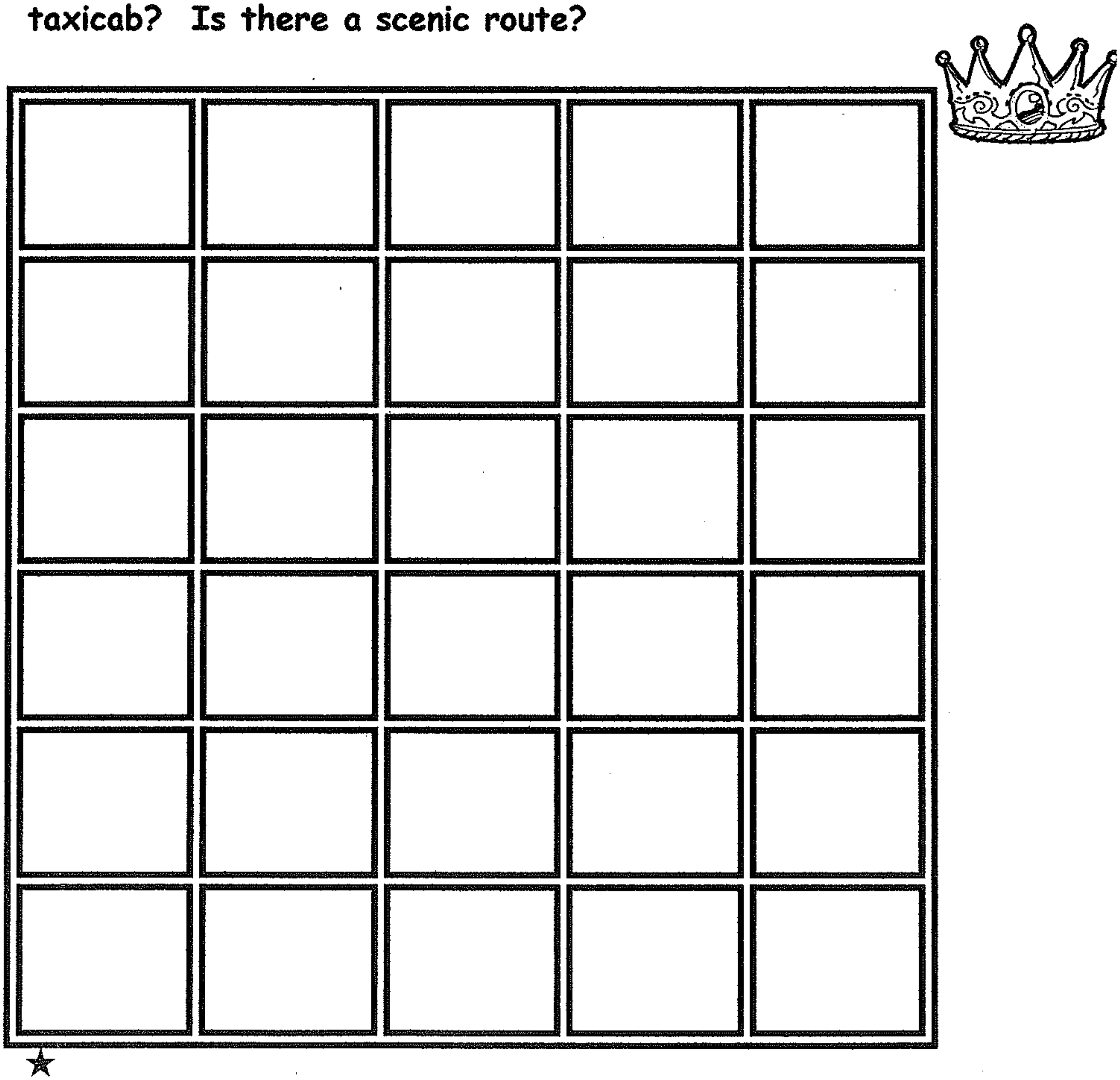
February, 2006:

Taxi cab Geometry: To launch the lesson, I put a new seating chart on the overhead and had students plan their "route" to their new desk. I then gave them 15 seconds to get to their new seat. Some students followed a path that was relatively linear while others went out of their way to avoid the crowd. We talked about where in life you would need to plan your route but also have a backup and also what businesses would be likely to use this. This introduced the idea of taxicab geometry. I then put up the overhead where they are summoned to the queen of Loprestiland.... We talk about how many routes are possible and if there is one that may be shorter. We then move to the superhero-land overhead where they can fly, thus making the route shorter. We then talk about how that might be measured which leads us to the Pythagorean theorem.

Again, my students are accepting the fact that there exist other geometries in which the idea of distance is paramount. They really enjoyed this activity and enjoyed the fact that even though many of them had heard of the Pythagorean theorem, few of them understood how it worked and how it could be related to another math topic. Again, students continue to "wrestle" with the ideas of non-Euclidean ideas. I believe that they understand that you can "change" the concept of what a line is but have a harder time believing the consequences of this change.

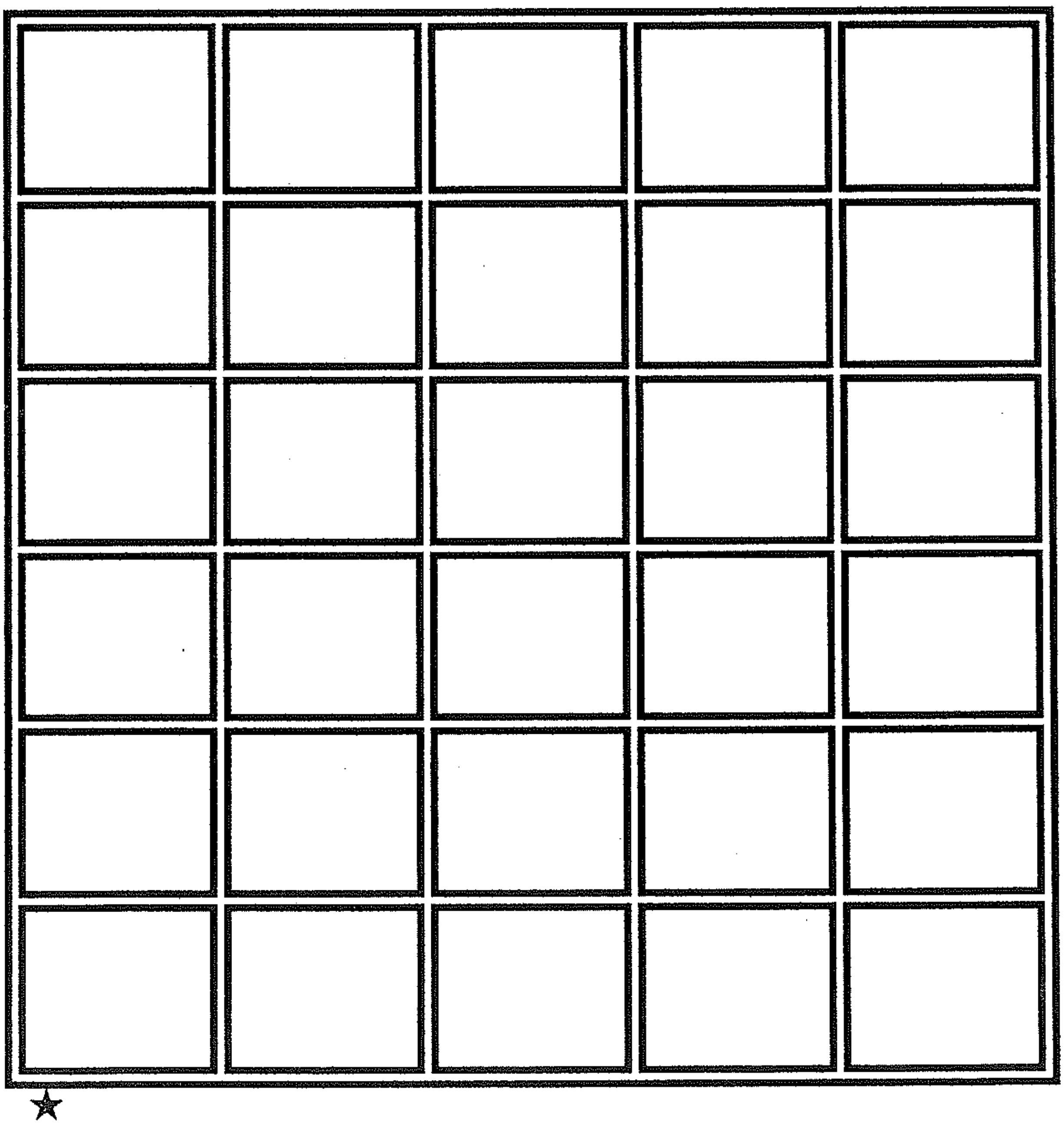
Taxicab Geometry!

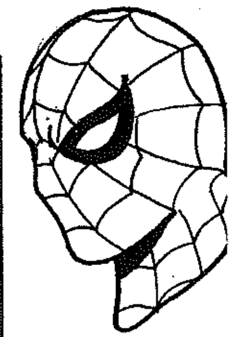
You are visiting Lopresti-land where all streets run either North to South or East to West. You have been summoned to appear before the queen. What are the different routes that you may take to get there by taxicab? Is there a scenic route?



You are here.

Now you have decided to visit the other Lopresti-land known as Super-hero-land. Again, all the streets are the same as the "real" Lopresti-land. Since you are in super-hero-land, you can now fly anywhere that you want to go. What route would you take now?

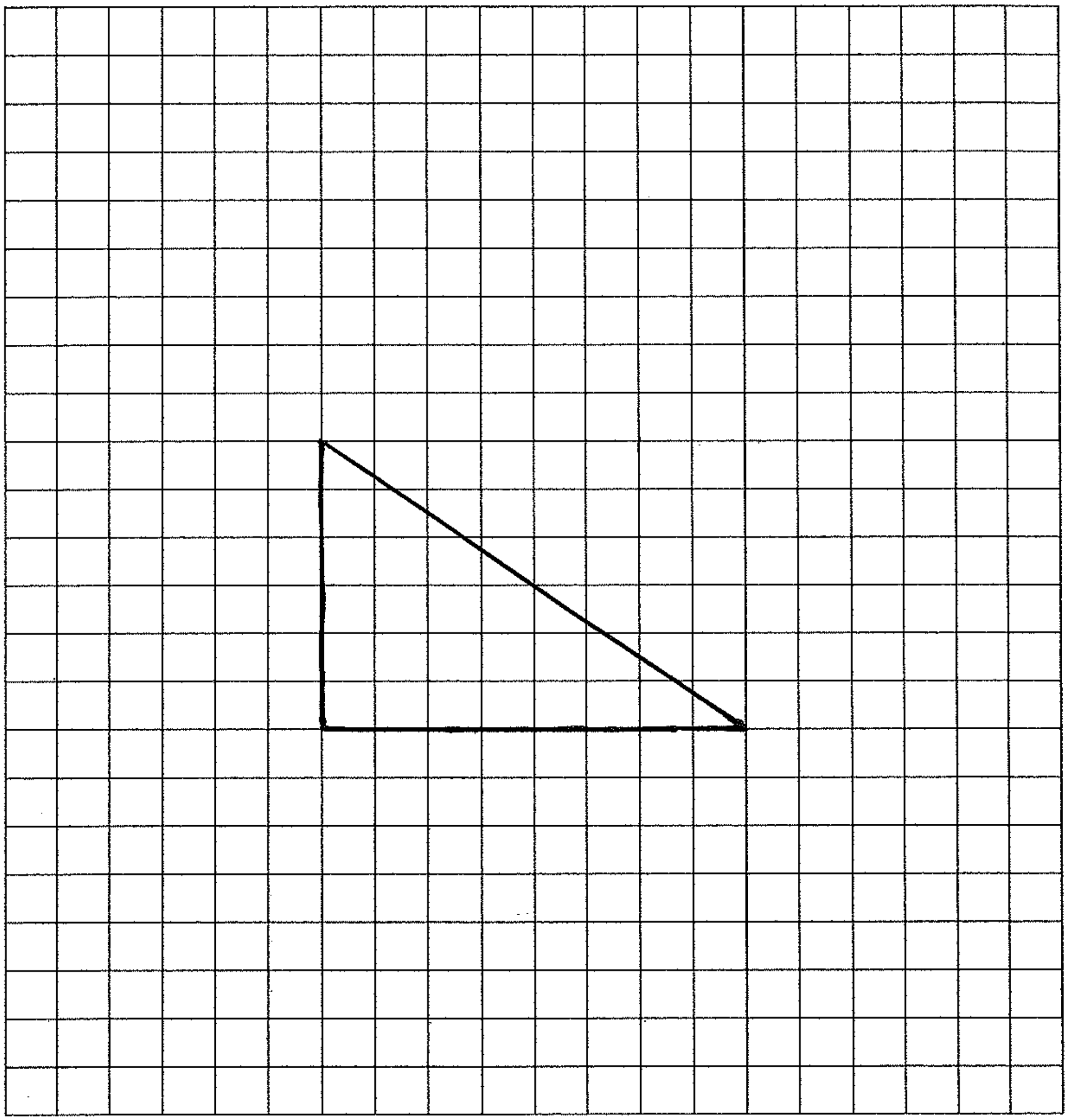




You are here.

So, how can we prove the Pythagorean Thoerem? There are over 300 proofs in existence!

Let's look at the drawing below. What do you notice?



Make squares on each side of the triangle. What would you do now????????

Activity 2

Content Objectives:

- Students will gain an understanding of how the idea of perspective has changed throughout time.
- Students will also be able to identify how the idea of perspective was influenced by the rulers of the time period.
- Students will be able to create a tile floor in correct perspective using the Alberti 3-stage method.

Language Objectives:

- Students will learn, use, and identify the vocabulary involved in perspective: perspective, horizon line, vanishing point, and orthogonals.

Supplies:

Note guide packet

In-focus projector

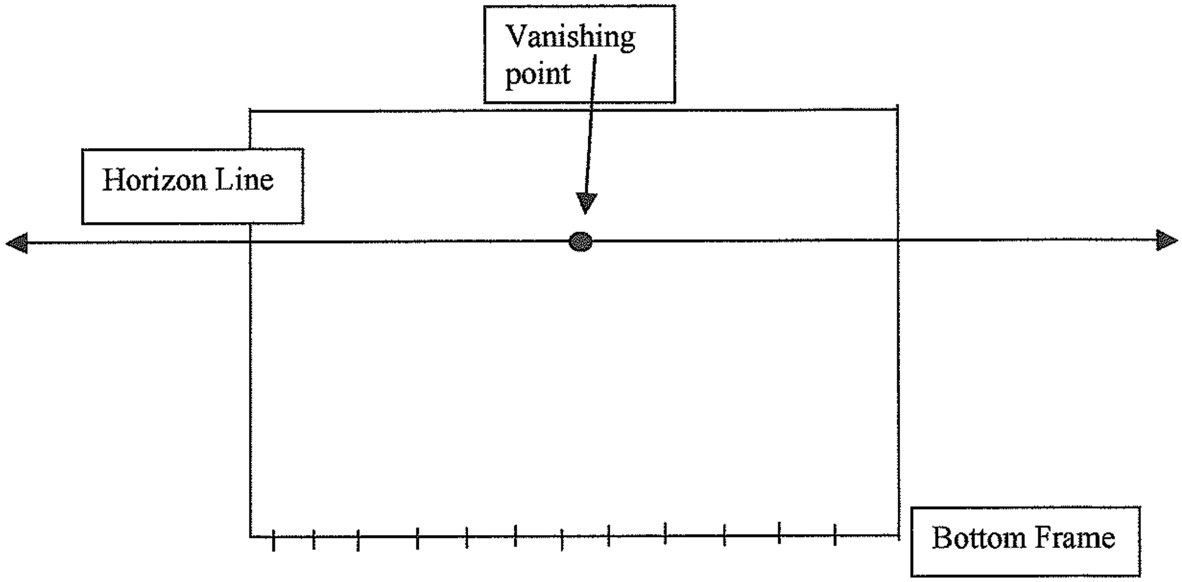
Power Point presentation

Rulers and paper for tile floor drawing

While presenting material in the power point presentation, students follow with note taking in the note guide. The power point presentation is included. Day 1 of the presentation involves art history only in regards to the use or lack of perspective. Students become familiar with the concept of perspective and how it has changed through different time periods. When presenting the Alberti 3-stage method, students will also create a tile floor in correct perspective. During Aleberti 3-stage method, I introduced the vocabulary of horizon line, vanishing point, and orthogonals.

Step for completing the Alberti 3-stage method:

1. Have students draw a horizon line, vanishing point and "bottom" frame on their paper. They will then measure equal distances across the bottom frame. I would recommend no more than 10 markings. They can then complete the frame by imagining a painter's canvas or a window as shown.



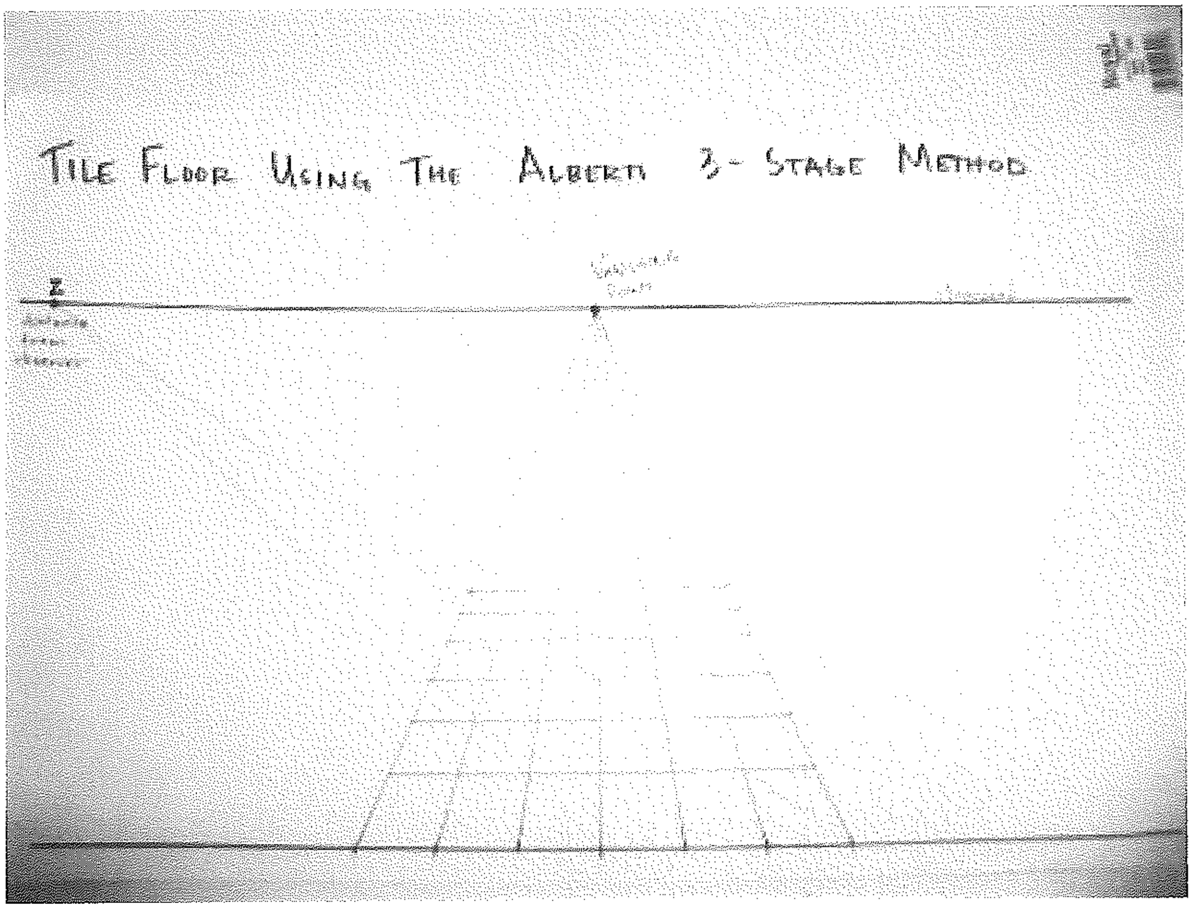
Note: The vanishing point and horizon line can be placed anywhere on the paper. However, I would encourage students to place them in the middle of their paper. That will make things much easier for students.

2. They will then draw a line connecting the vanishing point to each measured mark. This will create the beginning of a tile floor. These are the orthogonals.

3. Next, students need to pick a point Z which represents how far the viewer is away from the tile floor. The more distance between Z and the vanishing point creates a more extreme perspective. Point Z can be on either side of the frame.

4. Now, have students draw lines from point Z to the measured marks on the bottom frame. This creates where the horizontal lines will be drawn to create the tiles.

5. Where step 4 lines and step 2 lines cross is where the horizontal for the tiles needs to be drawn. This is the most challenging part of the drawing. A finished example follows.



This activity took longer to present than I had anticipated. My students struggled with note taking in regards to choosing the most important things to write down. I was also surprised by how many of them had trouble using a ruler and lining up points to create a line. The most surprising thing that happened during this lesson was how many of my "more reluctant" learners enjoyed this different format and less "mathy" topic.

MATH AND ART:

A PERSPECTIVE INTO PERSPECTIVE

THE HISTORY OF PERSPECTIVE
THE THEOREM OF DESARGUES

BY SHELLEY LOPRESTI

WHAT IS PERSPECTIVE?

PROFESSOR PERSPECTIVE WILL SHOW YOU...IS IT ART? IS IT MATH? OR IS IT BOTH??? TAKE A JOURNEY WITH THE PROFESSOR AND FIND OUT.

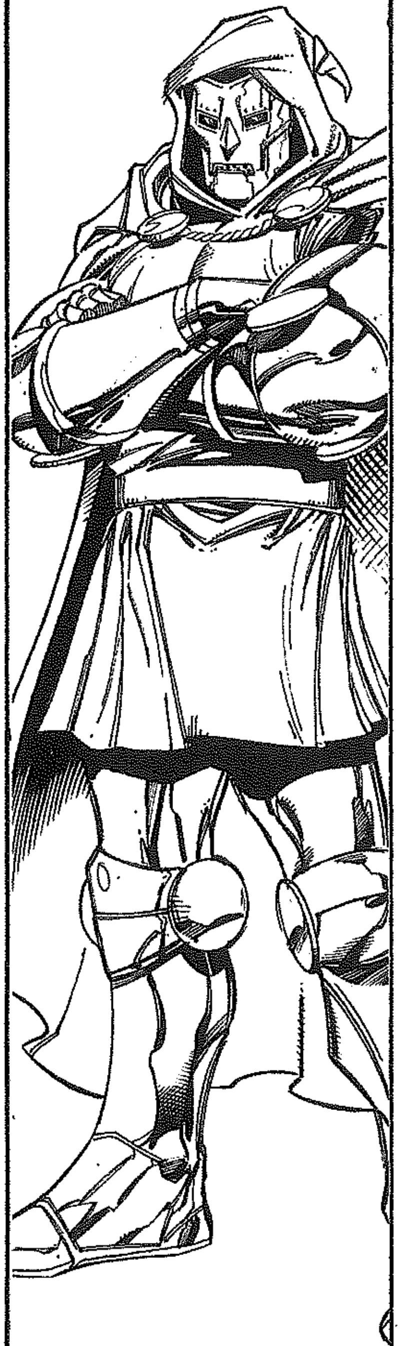
INTERIOR ART BY AARON LOPRESTI WITH PERMISSION FROM MARVEL COMICS. PROFESSOR PERSPECTIVE CREATED BY AARON AND SHELLEY LOPRESTI.

ART HISTORY

EGYPTIAN

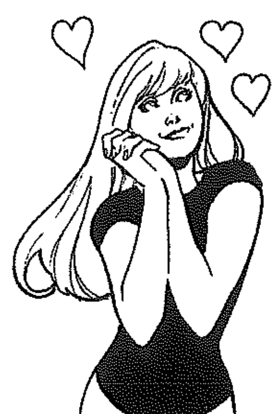
GREEK AND ROMAN

MIDDLE AGES



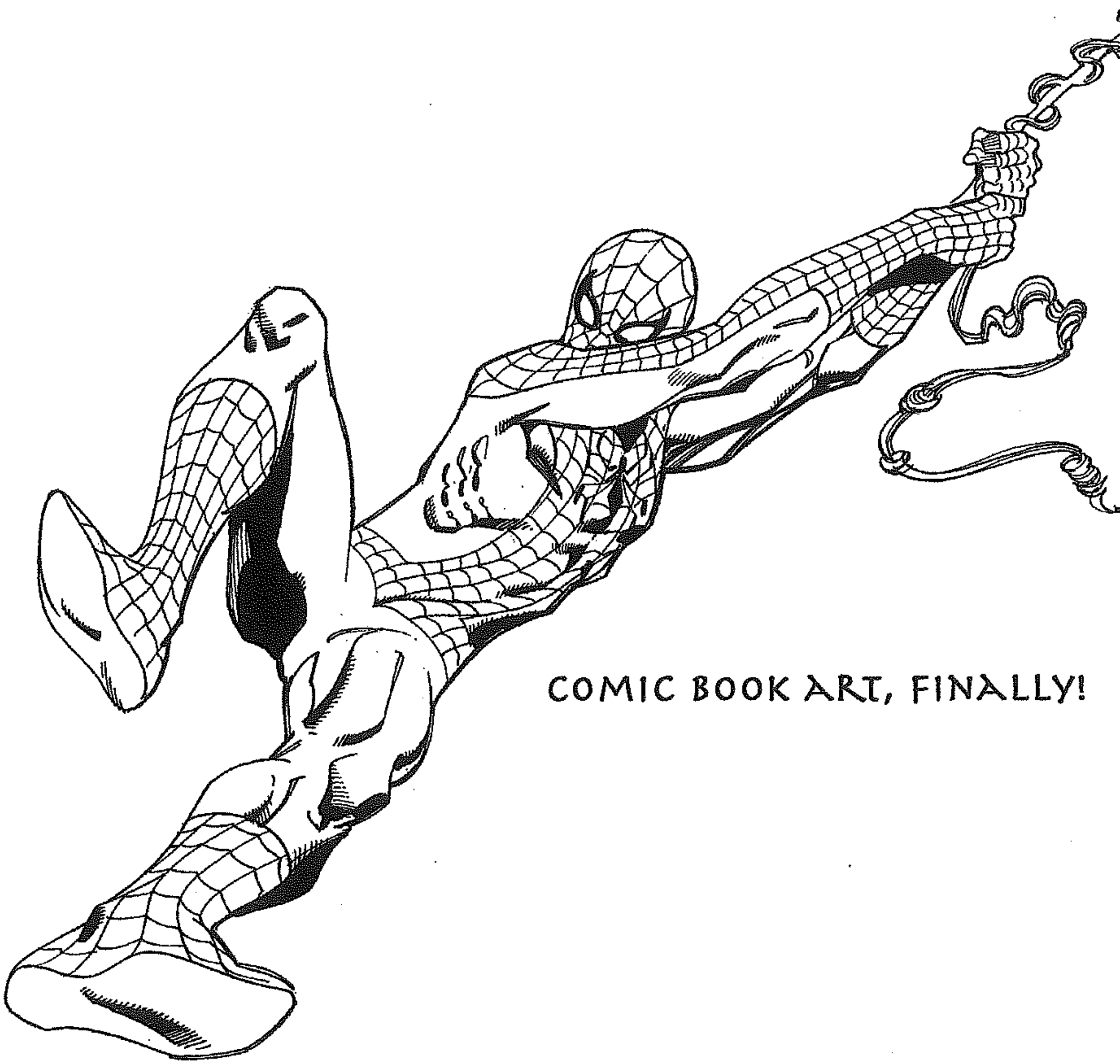
RENAISSANCE

ROMANTIC



MODERN ERA

MC ESCHER



COMIC BOOK ART, FINALLY!

MATH HISTORY

ROOTS OF GEOMETRY

LOGIC

GOLDEN AGE

EUCLID

COMMON NOTIONS

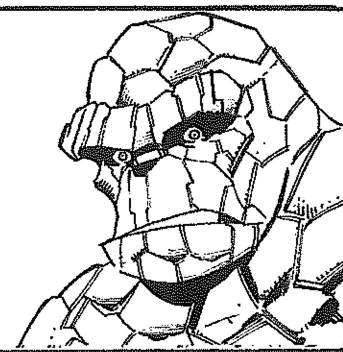
DEFINITIONS

AXIOMS OR POSTULATES

1.
2.
3.
4.
5.

THEOREMS

NON-EUCLIDEAN



GAUSS

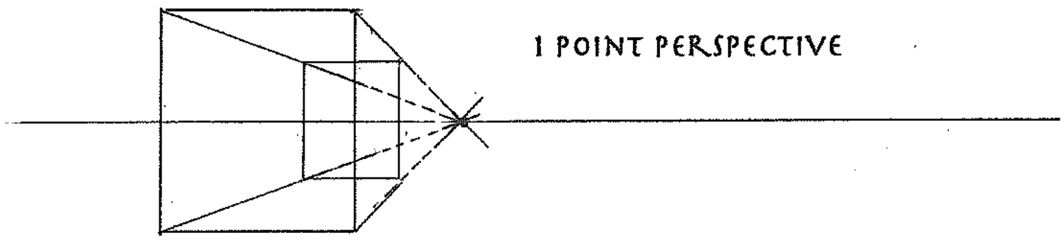
RIEMANN

GIRARD DESARGUES

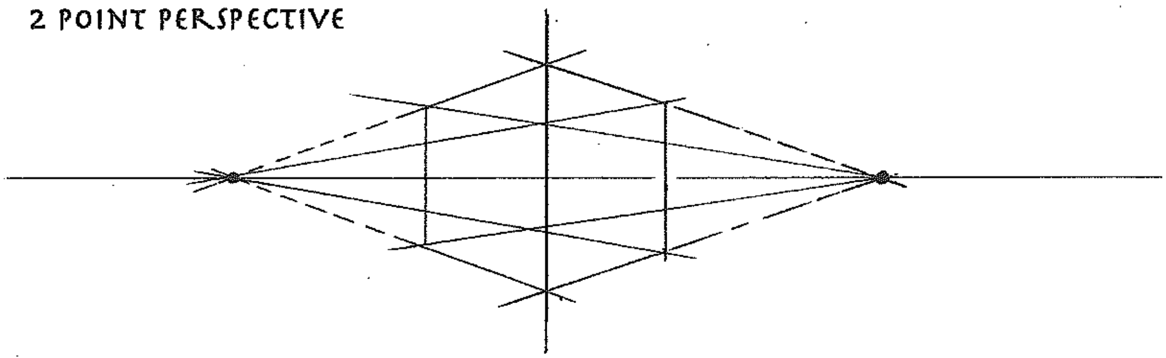
PROJECTIVE GEOMETRY

VALID AND CONSISTENT

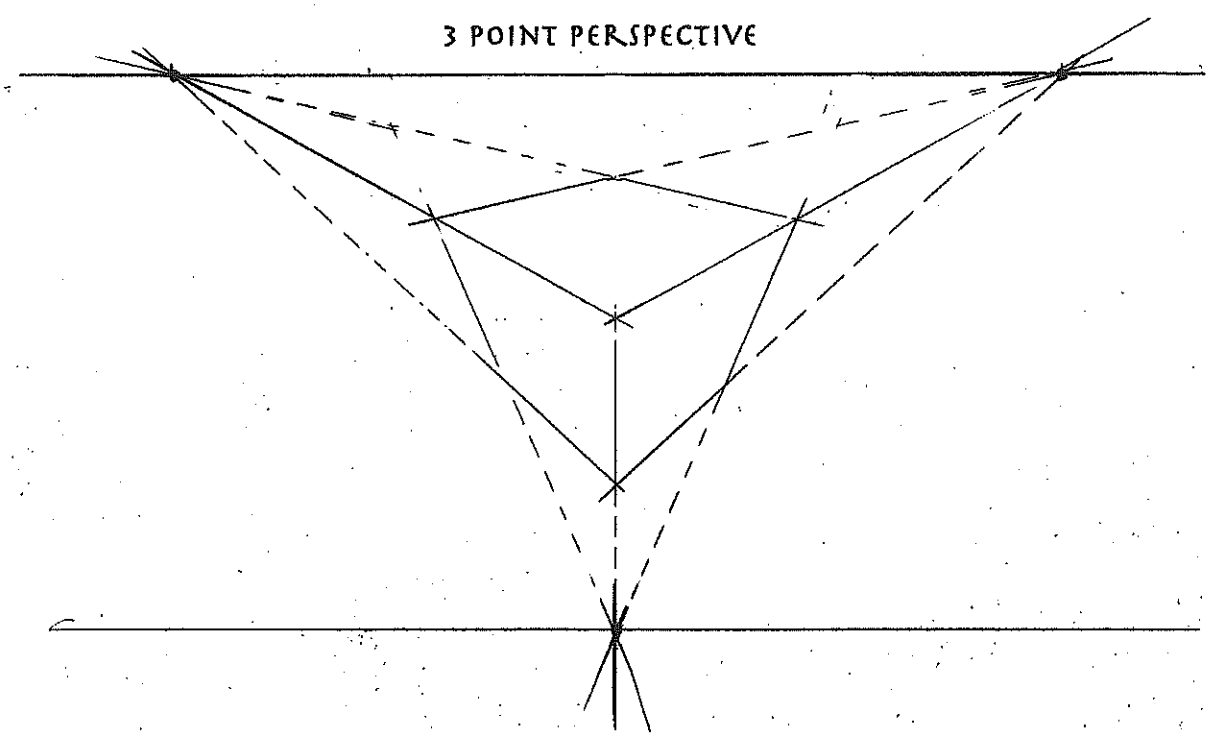
1 POINT PERSPECTIVE



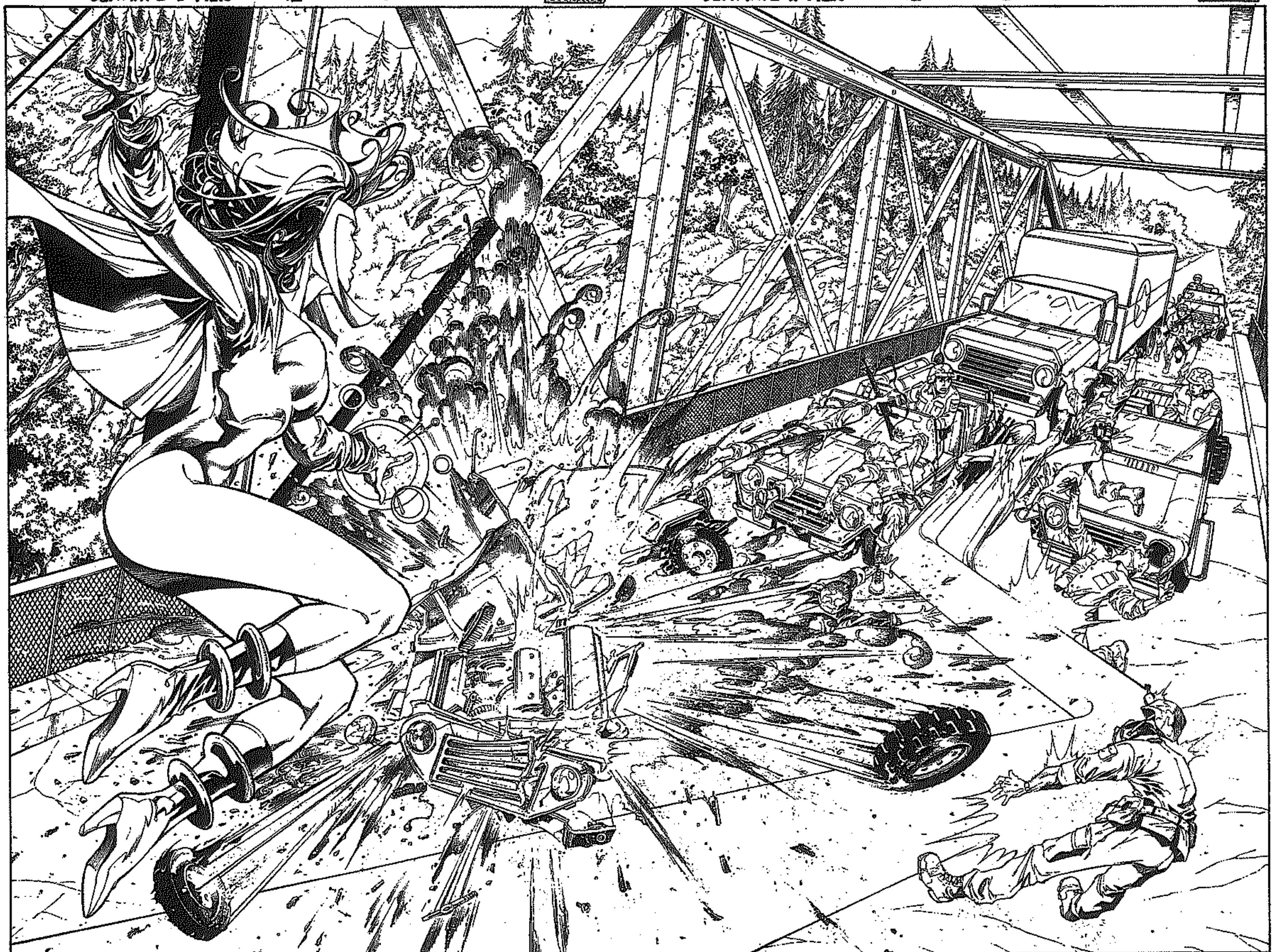
2 POINT PERSPECTIVE



3 POINT PERSPECTIVE



ULTIMATE X-MEN ½ 2 ULTIMATE X-MEN ½ 3 LOPRESKI · MIKI



CAN YOU FIND THE VANISHING POINT?


USMC
USMC
USMC

Activity 3

Day 2 of power point

Continue with power point presentation with the topic of math history.

Content Objective:

- Students will learn the history of geometry and how it is organized.

Language Objective:

- Students will learn and use the vocabulary involved in geometry.

Vocabulary: axiom, theorem, postulate, undefined terms, common notions

Supplies:

Power point with In-focus projector

Student note guide

During lesson, it is imperative to stress the point of the organization of a system. This will help students to make the transition from the Euclidean plane to a non-Euclidean system.

The following terms are used in all geometries and will help students to compare the different systems.

Definitions:

Undefined Elements: things that are accepted without formal definition.

Axiom or Postulate: the relationships between these elements that are without formal proof. These can be thought of as the rules of the game.

Theorems: Following from these axioms, there are relationships that can either be proven or deduced.

Definitions: relate to all the above-mentioned terms and relationships.

The same issues continue with this day of the presentation. However, I was more prepared to deal with the question, "Do I need to write this down?". Since this was not the first time my students have had the opportunity to look at organizing a system, I felt that they would be ready to embrace the new geometry in their future.

Activity 4

Day 3 of power point presentation:

Finish power point presentation with projective geometry and the theorem of Desargues.

Content Objective:

- Students will draw the theorem of Desargues with the center of perspectivity and line of perspective

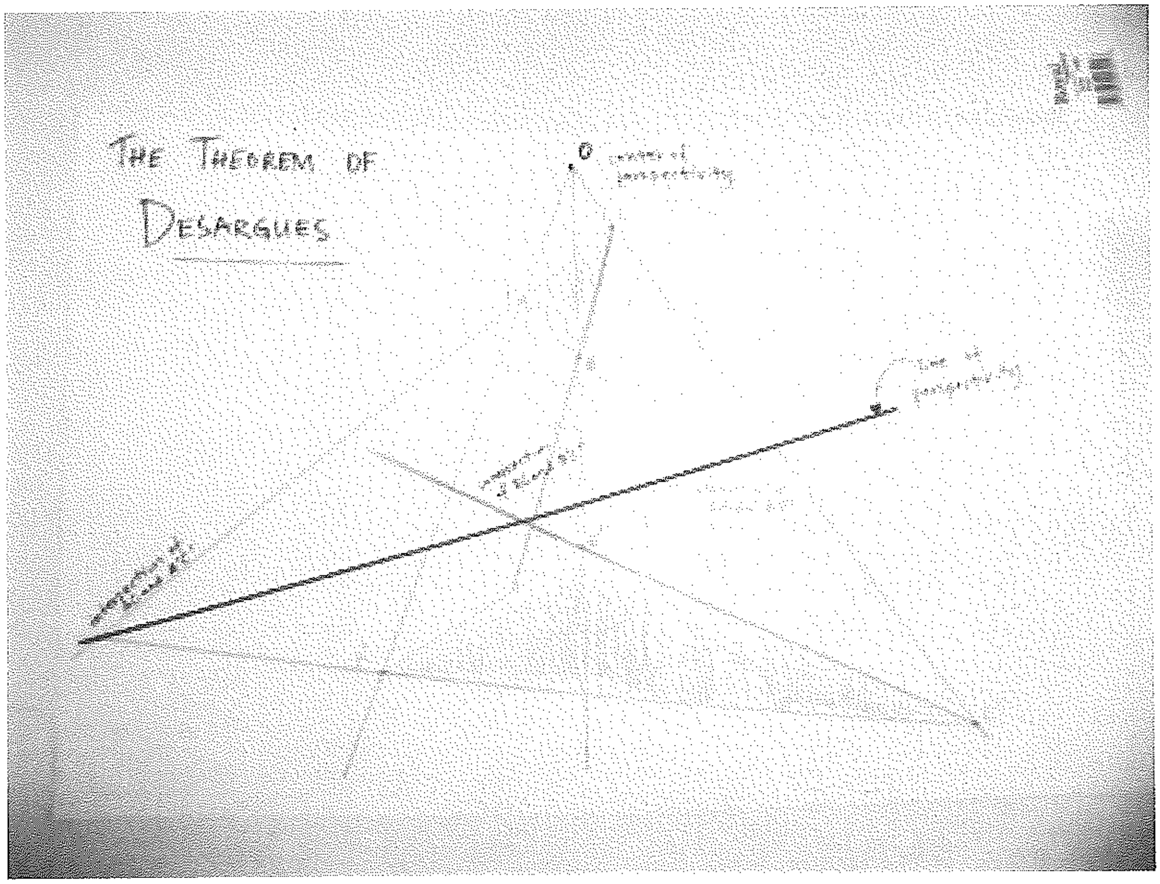
Language Objective:

- Students will learn and use the vocabulary of projective geometry and perspective.

Vocabulary: center of perspectivity, collinear, line of perspectivity

The following instructions will help students to draw the theorem of Desargues.

1. Have students establish their center of perspectivity or point O. Students then need to draw 3 lines that are concurrent (meet at) O.
2. Students will then create two triangles perspective from O by establishing corresponding points that are collinear. For example, on the first line, they will make points A and A'. On the second line they will create B and B' and on the third, C and C'.
3. Students create triangles ABC and A'B'C'. This sets up our theorem, the "if" part. Students need to be able to "see" the two triangles and that they are in perspective from point O.
4. Next, students need to extend the sides of the triangles. First have them extend AB and A'B'. These will intersect. What students will begin to realize is that placement of those points is very important if they want their lines to intersect on their paper or in another town... This will continue with AC and A'C', then with BC and B'C'.
5. The three new point will be R, S and T. These points will be collinear thus illustrating the theorem.
6. This new line is the line of perspectivity.



The power point presentation today involves the building blocks of projective geometry. My goal for this day is to have my students gain an appreciation that perspective is actually mathematically based. Students will also draw the theorem of Desargues. This day concluded the power point presentation. By the end of this unit, each student will have a completed note guide including a tile floor drawing (Alberti 3-stage) and an illustration of the theorem of Desargues. This is part of their grading for this project.

Activity 5

Day 4:

Today, we will focus on the idea of projection using map projections.

Content Objective:

- Students will understand the idea of projection in making maps. They will experience how different projections will distort or accurately depict map distances or areas.

Language Objective:

Students will learn and use vocabulary used in map projections.

Supplies:

12 inflatable world globes

12 different flat maps (mercator, Peter's projection, upside down world map)

1. Students will get into groups of 3.
2. As a group, they are to list the 7 continents in order from largest to smallest in terms of area. Answers: Asia, Africa, North America, South America, Antartica, Europe, Australia. Be aware that there are different ideas of the number of continents. For example, in some cases the islands of the Pacific are referred to as Oceania instead of Australia.
3. As a class, go over their lists to come to a consensus.
4. Pass out flat maps and have them recheck their lists. At this point, students also look at attributes of their map: legend, what type of projection and how distance is measured. As a group, have students compare the area of Greenland to Africa. (in reality, it is 14x smaller)
5. Pass out inflatable globes and again check their lists.

   Students at this point start to notice that relative areas are different.
6. As a group, have students compare and contrast their flat map to the globe. (t-chart acceptable)

7. Have groups share their flat map and what kind of projection it is. Students can see that different projections make a different looking map.

8. Go through overheads of different kinds of map projections and how they are done.

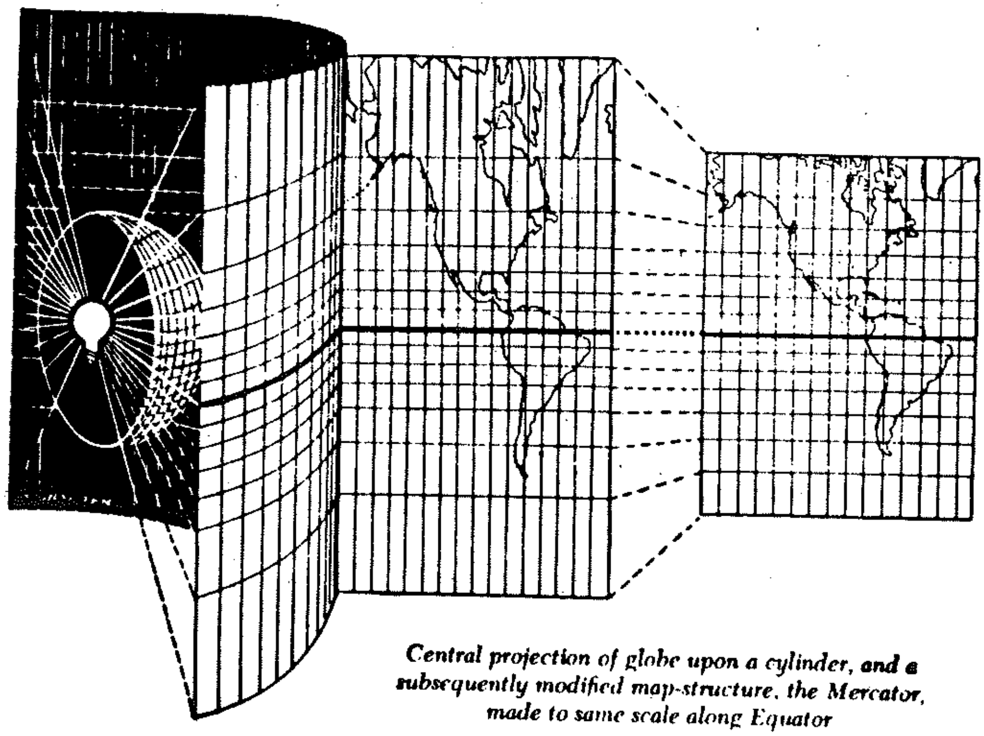
Reflection on map activity

I thoroughly enjoyed this activity. This activity was inspired from an activity given to me my Julie Keener from Central Oregon Community College. In researching mapping, I felt that an entire project could done on map-making. However, in my mind, the purpose of this activity was for students to experience another application involving projection, and of course to some, map making is artistic as well. The aspect I enjoyed the most was how my students responded to topics that, as they put it, "mess with our minds". This feeling of disequilibrium shows me that they are learning.

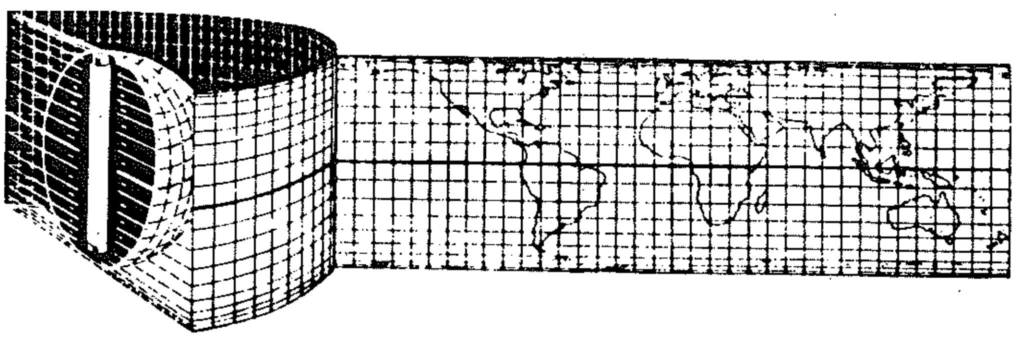
As stated in a book called Mapping by David Greenwood (1964), "Maps are supposed to give information, but they can also put up an argument. This argument will appear so convincing that only another map can successfully refute it." The challenge of making accurate maps is a problem that has been around as long as man. This challenge lies in the fact that some part of the map must lie. It is impossible to keep all aspects of the map within the bounds of being "truthful". In having my students make a decision about the relative size of the continents, they were invested in their beliefs. Of course, many were most familiar with the mercator projections where most of the distortion is at this poles as the geodesics through the poles are now parallel. When comparing the size of the continents using both a globe and a flat map, students were able to see the differences. Some of the maps, namely the Peter's projection maps, preserve area but distort distance. As one student commented, "This map is too skinny!". I am glad that I was able to find maps that were made from different constructions and that a colleague, Tom Arend, gave me some overheads that he used to use in his geography classes. Who knew that they would now belong to a math teacher?

These overheads gave my students a visual representation of different projections in map making. I, as well as my students, was surprised at how many different maps there are. When one student asked, "Why are there so many?", it was great when they all realized that it depended on what you needed it for. The mercator map, while misrepresenting the relative size of the continents, was actually the map of choice for navigation when all you were worried about was where you are.

In listening to students comments during this lesson, I was pleasantly surprised by their realization that math is everywhere. Also, hearing the word "cool" used by students in a math class was well worth the time and effort of putting this project together.



Central projection of globe upon a cylinder, and a subsequently modified map-structure, the Mercator, made to same scale along Equator



Shadowgraphing globe upon cylinder of same area. Note parallel rays (supposedly produced)

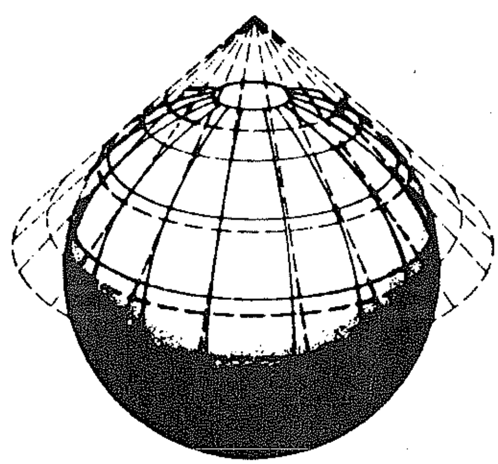
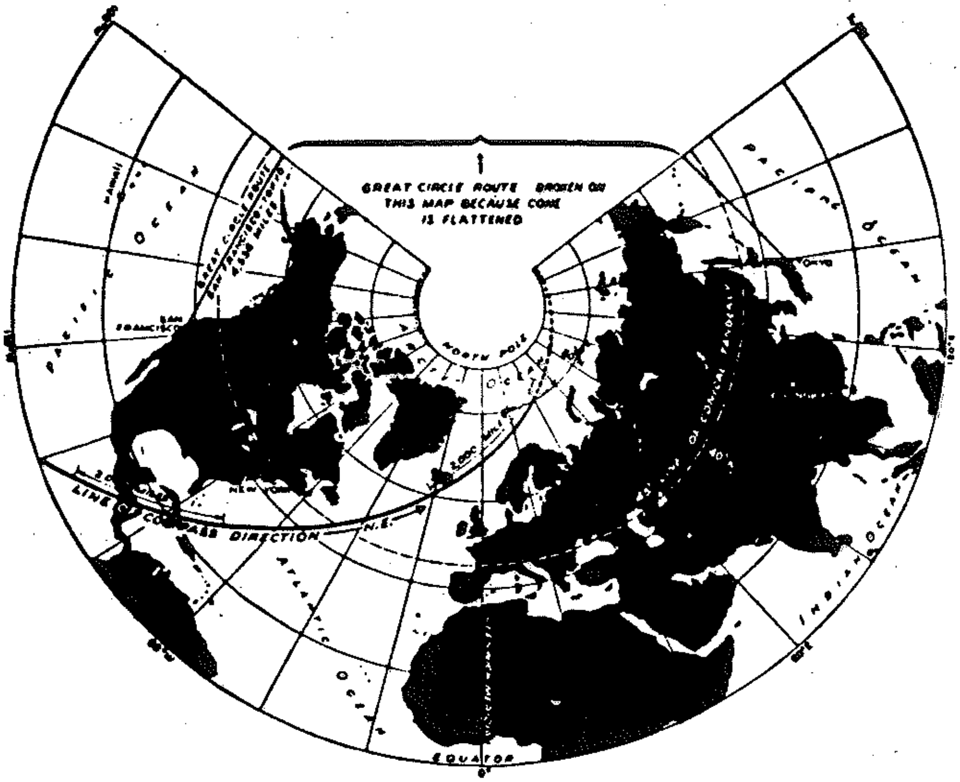
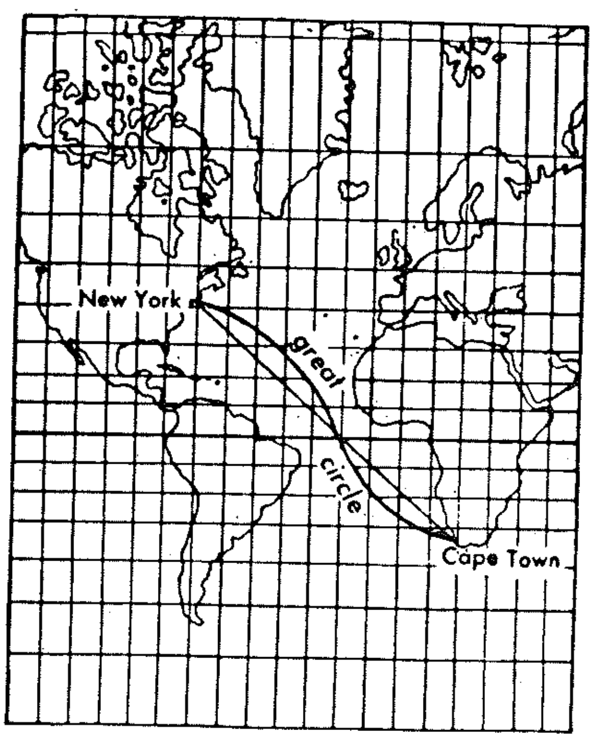


Figure 2.7 Conic projections use the principle of a cone resting on a sphere.



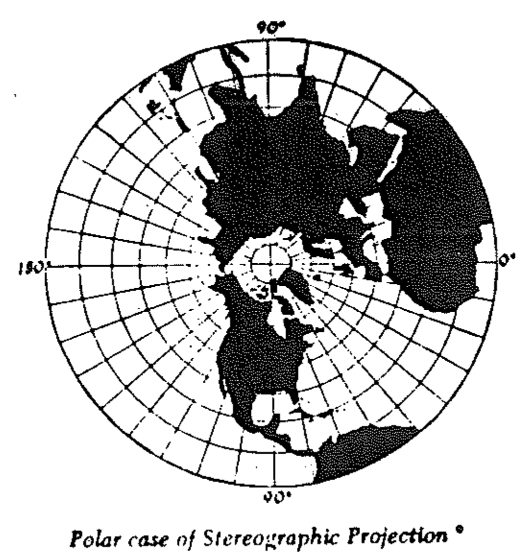
Full spread yoke of Simple Conic Projection. Note spiral of line of constant oblique direction •

Polyconic Projection, from Pole to beyond Equator



Rhumb-line and great-circle paths athwart the Equator on the Mercator

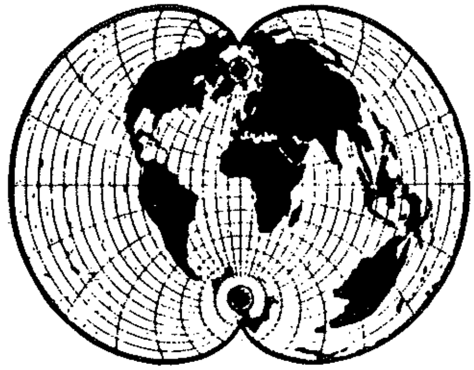
Equatorial case of Stereographic Projection



Polar case of Stereographic Projection •

Denoyer's Semi-Elliptical Projection †

Van der Grinten Projection of sphere within a circle



What happens when the Polyconic is developed to include the entire sphere

Sinusoidal Equal-Area Projection of sphere

Werner Equal-Area Projection of sphere

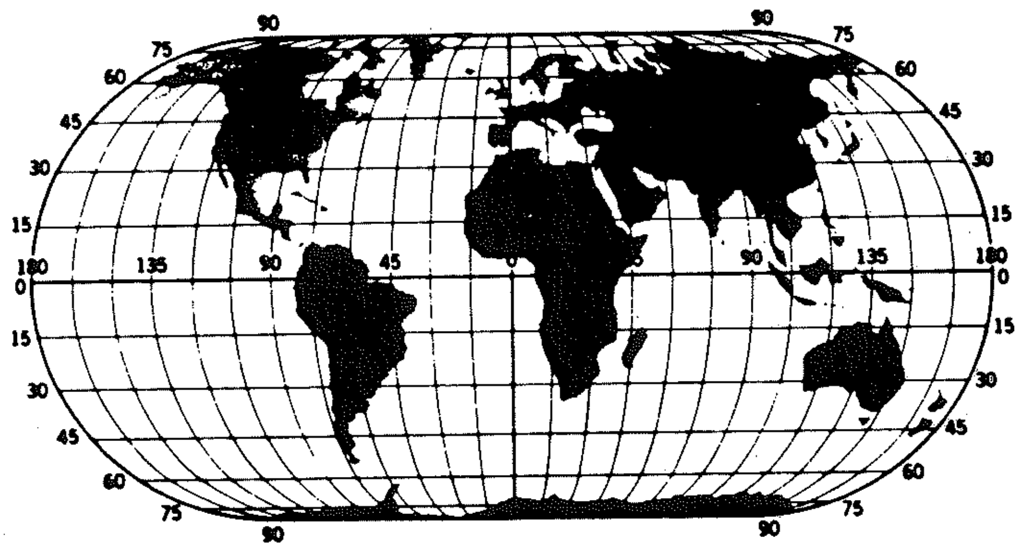


Figure 2.26 The Eckert IV projection uses horizontal lines to represent each pole.

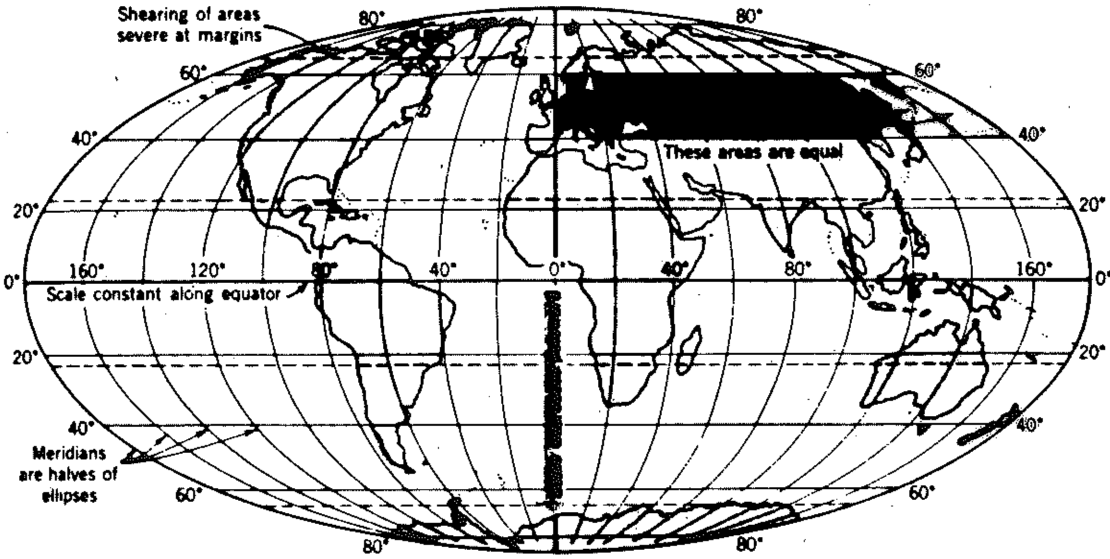


Figure 2.22 The Mollweide homolographic projection is widely used to show areal distributions over the globe.

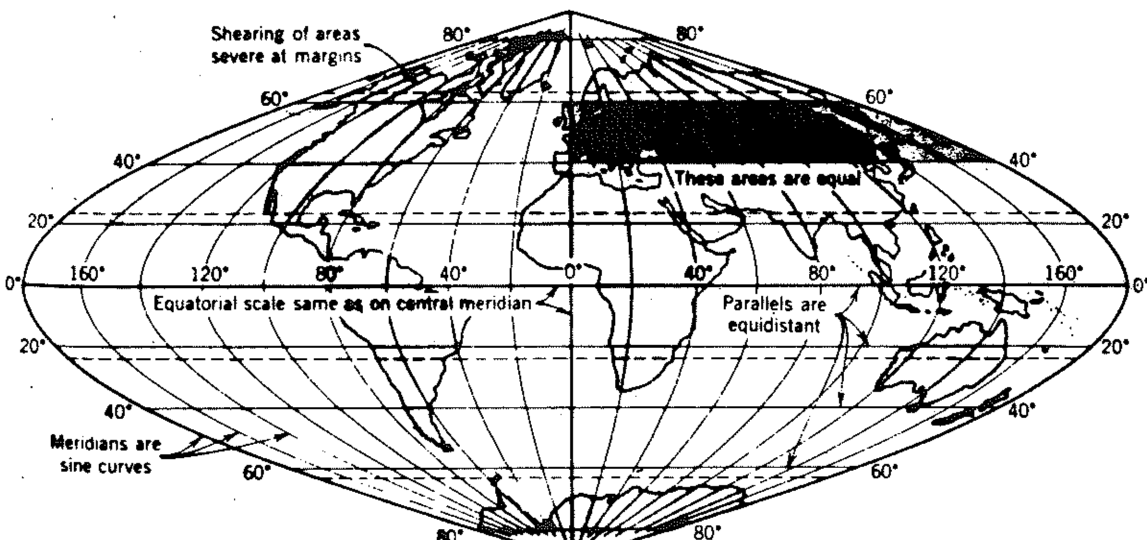
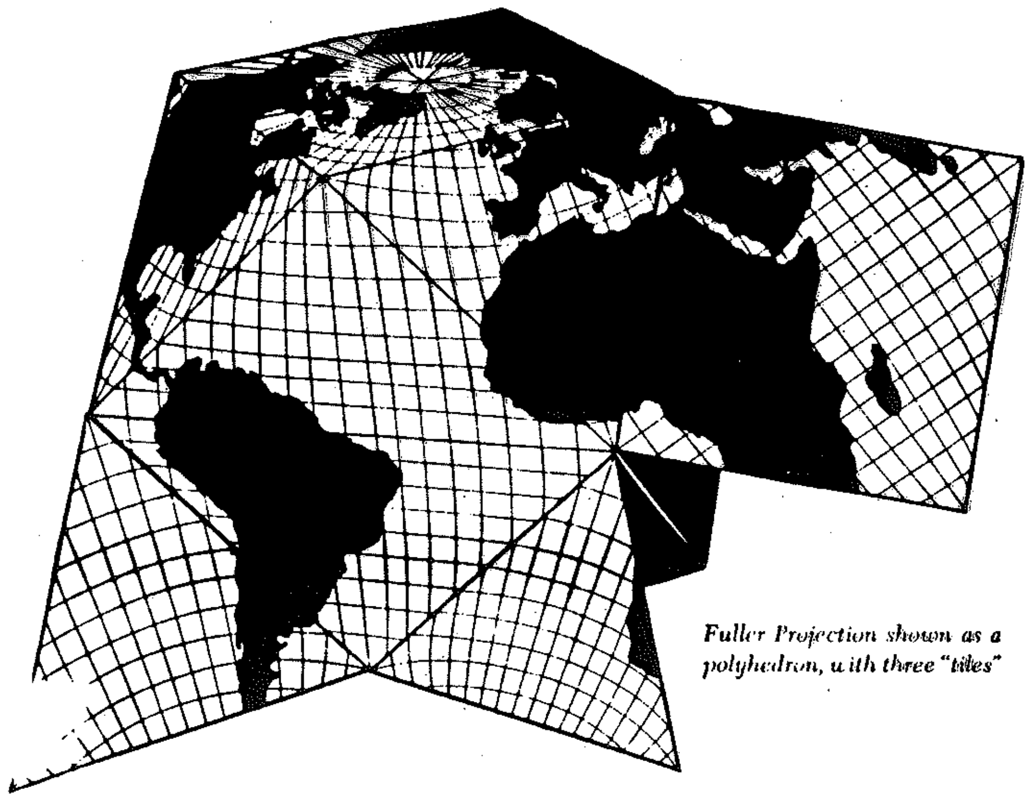
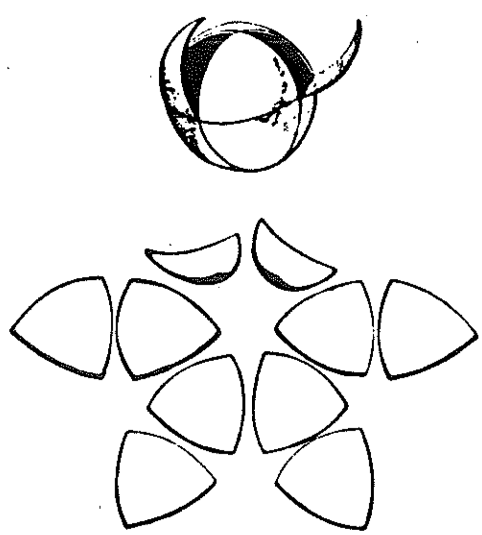
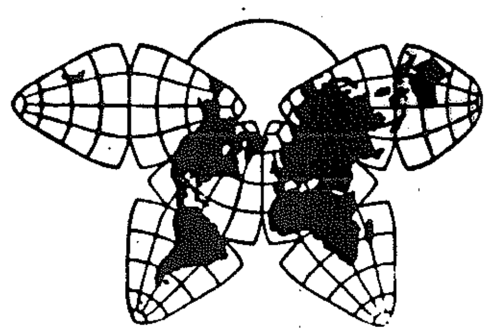


Figure 2.23 The sinusoidal projection is an excellent equal-area projection for lower latitudes.

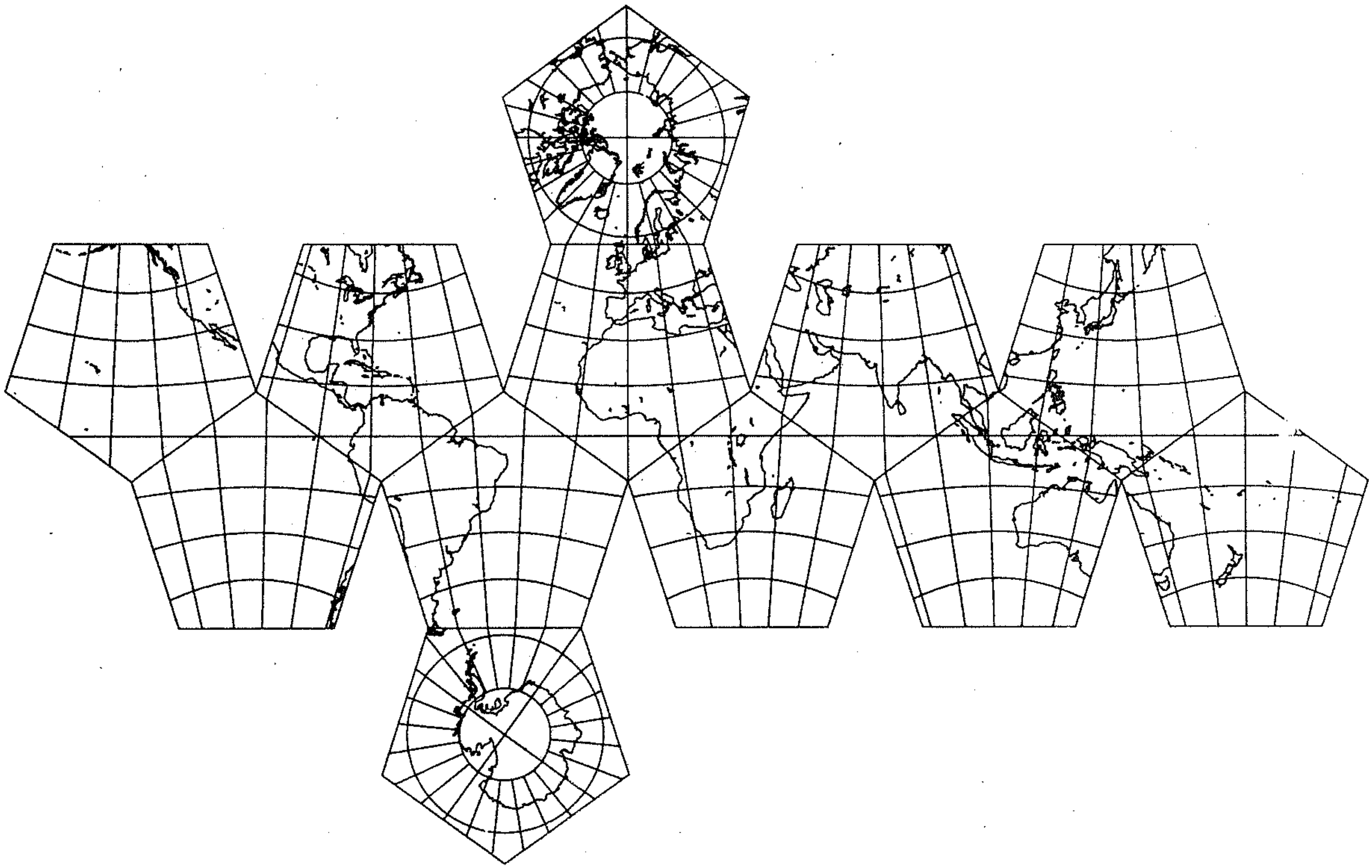


Fuller Projection shown as a polyhedron, with three "tiles"





Cahill's Projection *



Gnomonic Dodecahedron;
Polyhedral Globe; Platonic Solid;
Source: John Parr Snyder's Personal Software

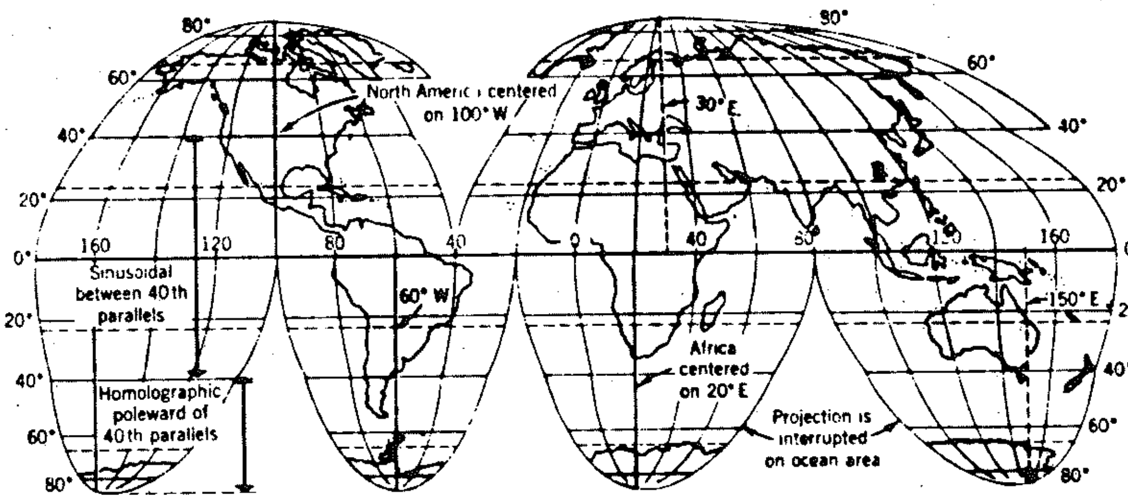


Figure 2.25 Goode's interrupted homolosine projection combines the homolographic and sinusoidal projections. (Based on Goode Base Map. Copyright by the University of Chicago. Used by permission of the University of Chicago Press.)

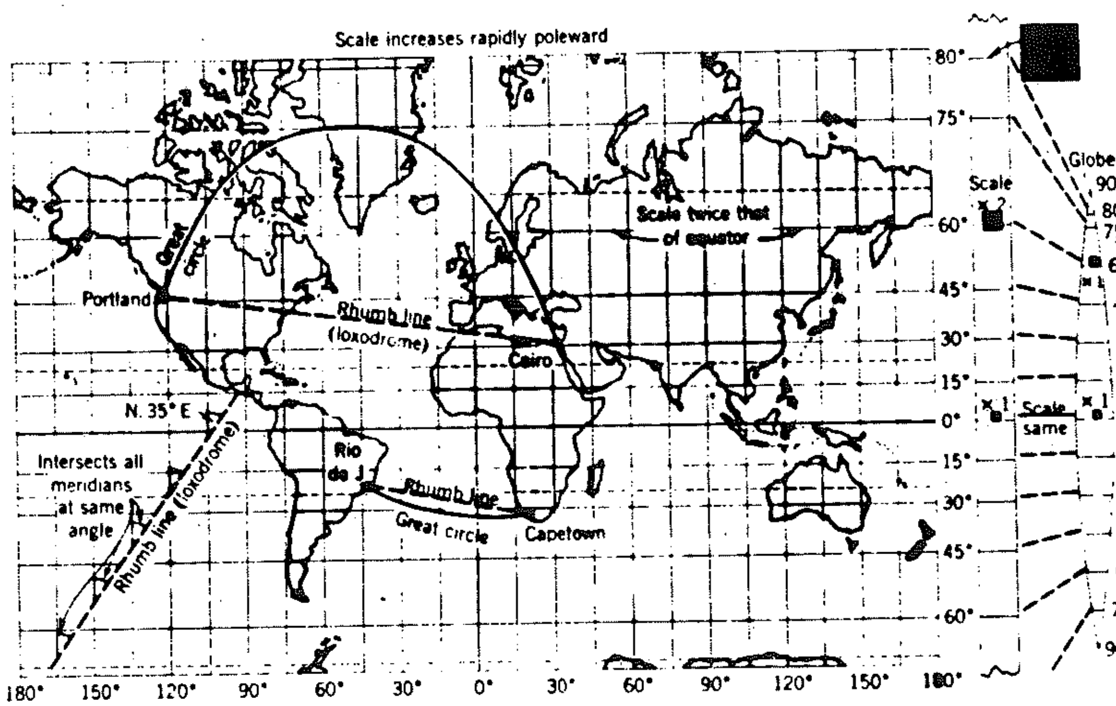


Figure 2.18 The equatorial Mercator projection shows all lines of constant compass direction as straight lines.

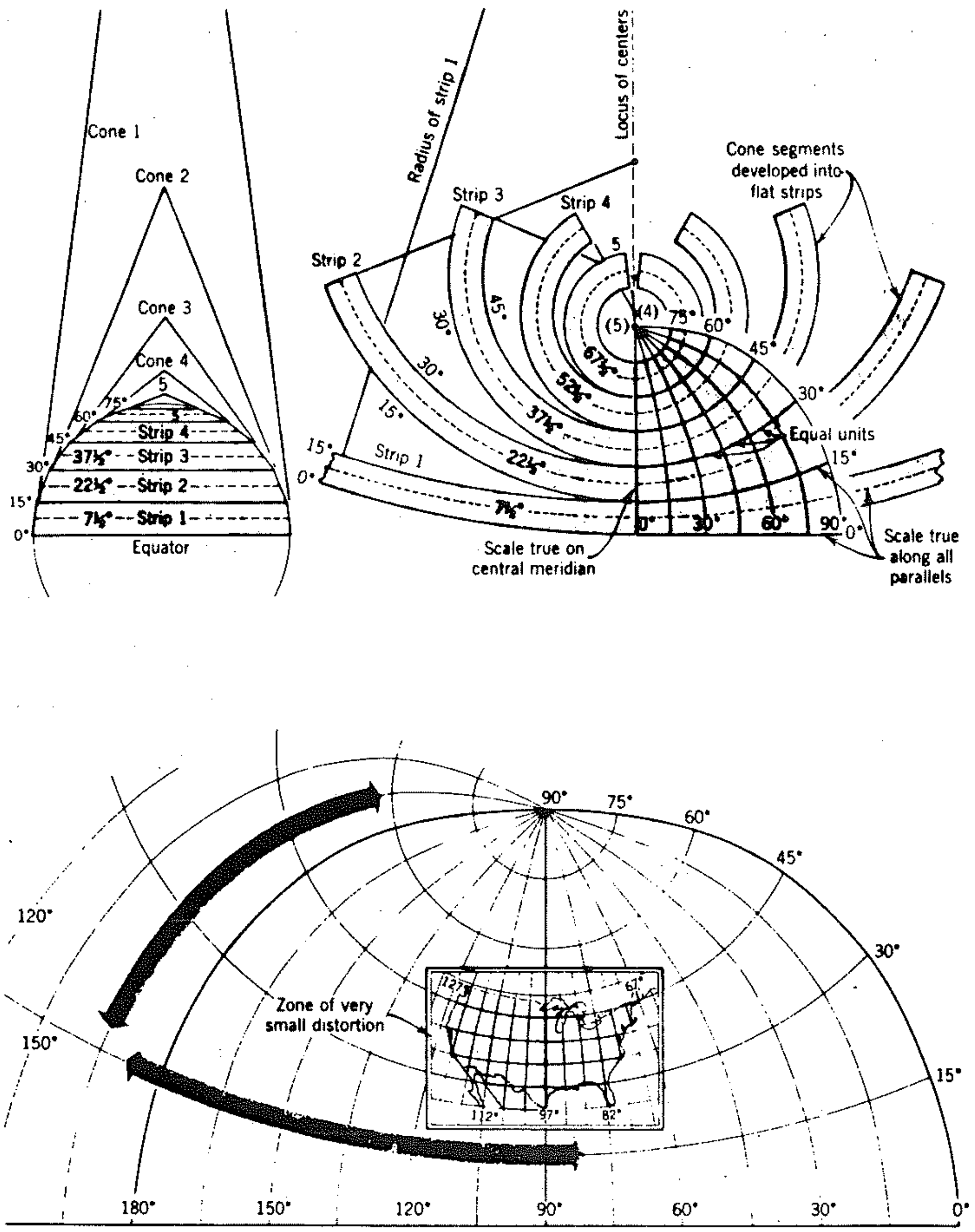
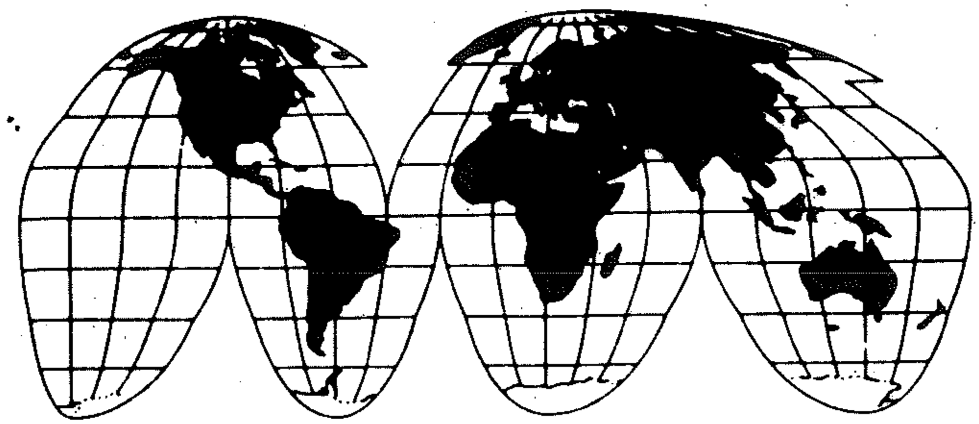
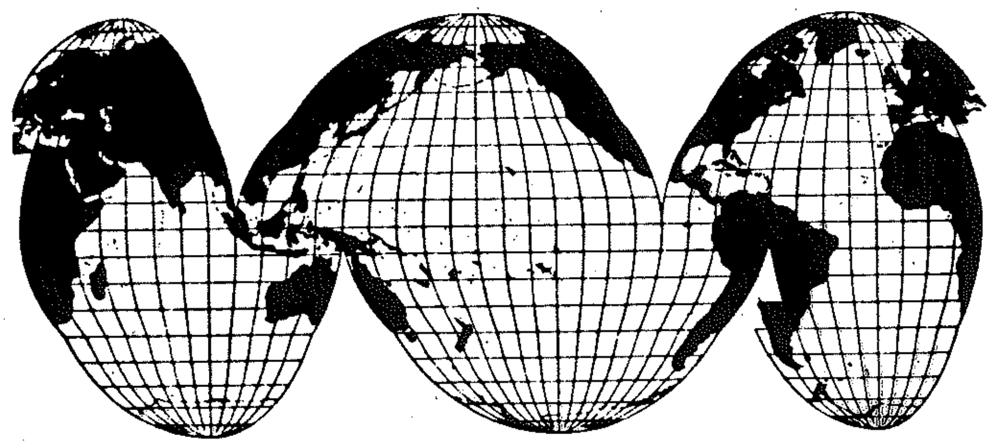


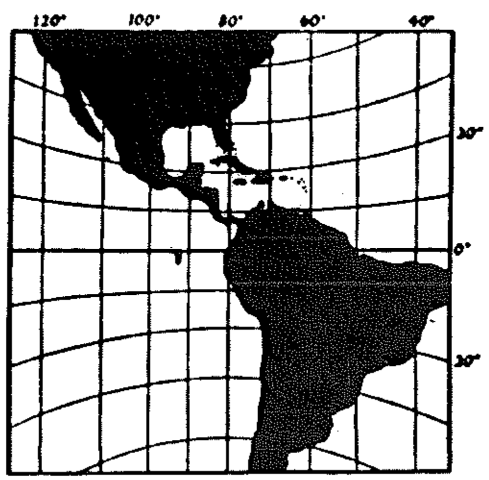
Figure 2.17 The polyconic projection is used for small areas in middle latitudes.



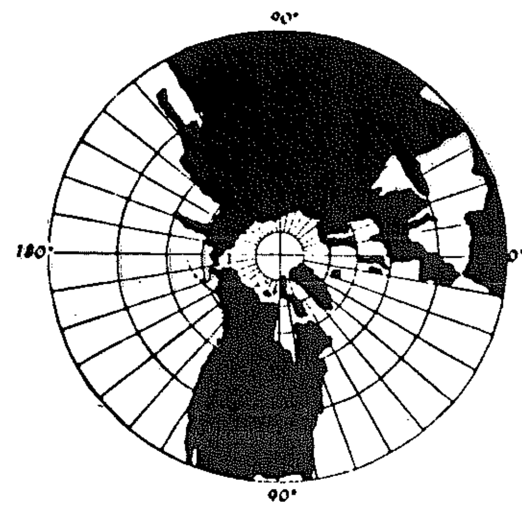
Goode's Homolosine Equal-Area Projection. Oceans interrupted to exhibit continental areas



Goode's Homolosine Equal-Area Projection. Continents interrupted to exhibit ocean areas*



Equatorial case of Gnomonic Projection, centered at 80° W †



*Polar case of Gnomonic Projection, extended to Lat 30° **

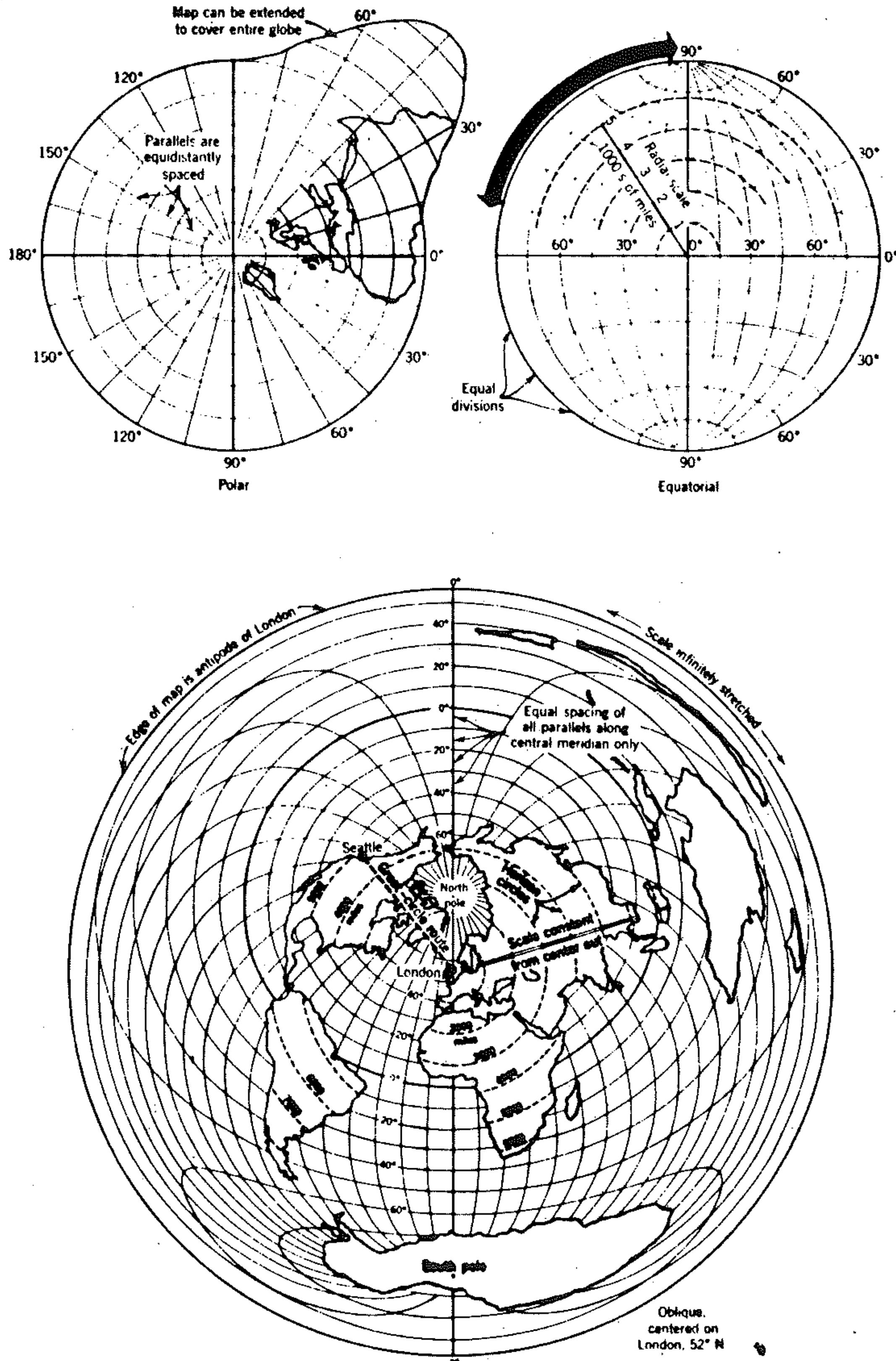
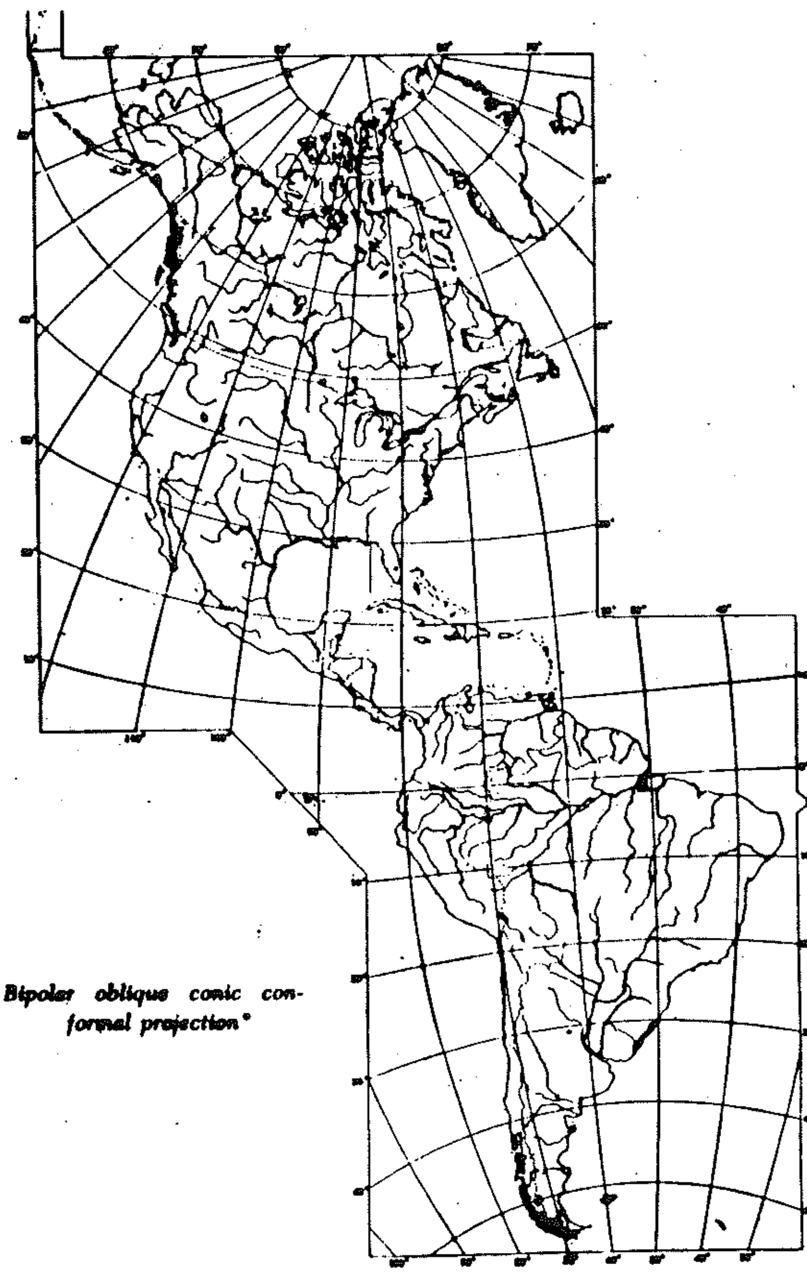


Figure 2.12 The azimuthal equidistant projection is useful in measuring distances from the center to other points.



*Bipolar oblique conic conformal projection**

Activity 6

Students will work on culminating project. Project should show an understanding of perspective either in a piece of artwork or mathematically by drawing the theorem of Desargues. Other acceptable projects range from photo collage, a perspective tool or a research project on either a mathematician or artist from the Renaissance.

In addressing the diversified ways that students learn, I have allowed students to choose their project. I have put very loose parameters on this project as I find that most students, when left to their own "devices", will do more than I would have expected. In this regard, I was not disappointed.

In looking back, I am very pleased with the quality of work from my students. The most interesting, and surprising event with this project is the quality of work from some of my "less motivated" students, as it was great. Many felt that this was a great change from the traditional assessment tools that they were used to. This was a pleasant surprise for me. I believe that projects are an excellent way to assess students' knowledge, especially when they get to choose their topic. The projects turned in ranged from written work on research from geometry, art or even map projections to a student constructing the Durer perspective aid from wood.

Perspective Project

You have several options for your final project in our math and art unit. This should show your understanding of perspective.

1. Draw something in perspective.

2. Create a photo or art collage of examples of perspective including the vanishing point.

3. A short paper on map projections, the history of perspective or a mathematician of the Renaissance.

4. A detailed drawing of the Theorem of Desargues. Bonus: Show how different points in the Desargues' Configuration will work.

5. Create a 3-D model of the visual pyramid including the Theorem of Desargues.

6. One of your own original ideas. Check with me first.

Due date is June 12, 2006 for all students. Bring them to A-10 in the morning.

This project will be graded on completeness (5 pts), creativity (5 pts), your understanding of the topic (10 pts.).

Each project will be worth 20 points for an assessment grade. Be creative!

Name:____.

Project:____.

Points:____________________/20

Final Reflection

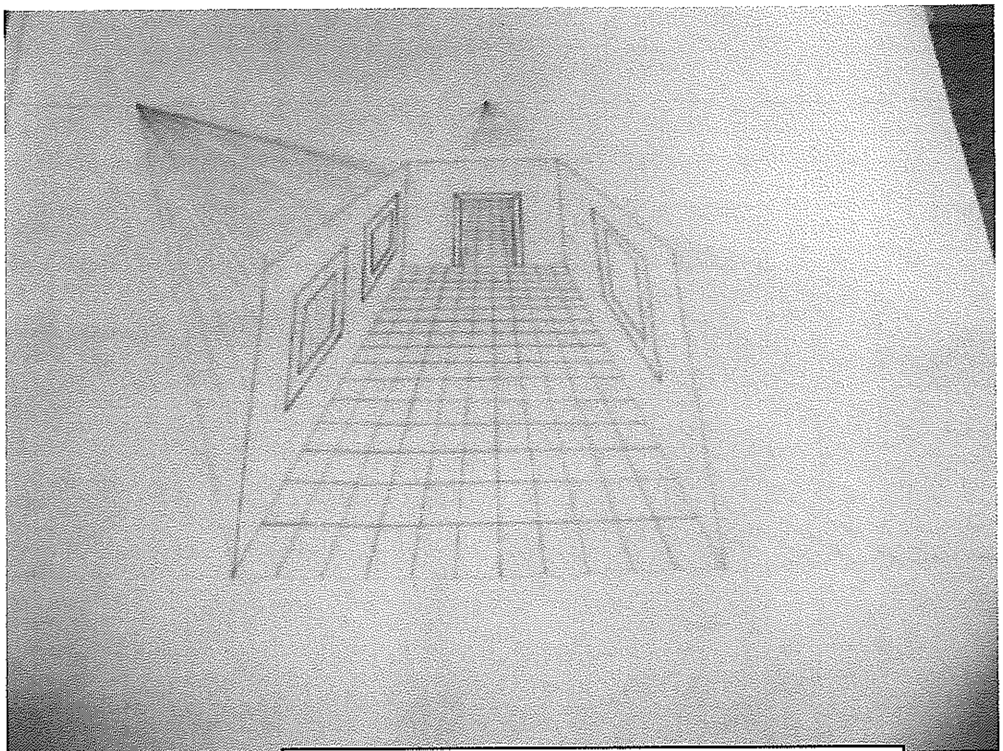
As I sit here at my computer trying to create a final reflection of my project, I must let out a sigh of relief. When beginning this project, I had no idea the "can of worms" I would be opening. I knew that I was intrigued by the concepts of non-Euclidean geometries but also wanted to incorporate a topic that students would enjoy. I think that this project was a nice combination of both. I definitely challenged myself in learning about projective geometry and art; both of which I knew very little.

I was concerned about how to convey all this information to my students in a way that they would find interesting but also addressed the learning styles of my students. I knew that if I communicated excitement for the topics and curriculum, my students would follow. I guess a sort of "Pied Piper" of mathematics. In looking back, students enjoyed the power point presentation. In examining artwork, it was great to hear students making statements and actually noticing what I wanted them to without being told. It is so nice when things fall into place and actually work.

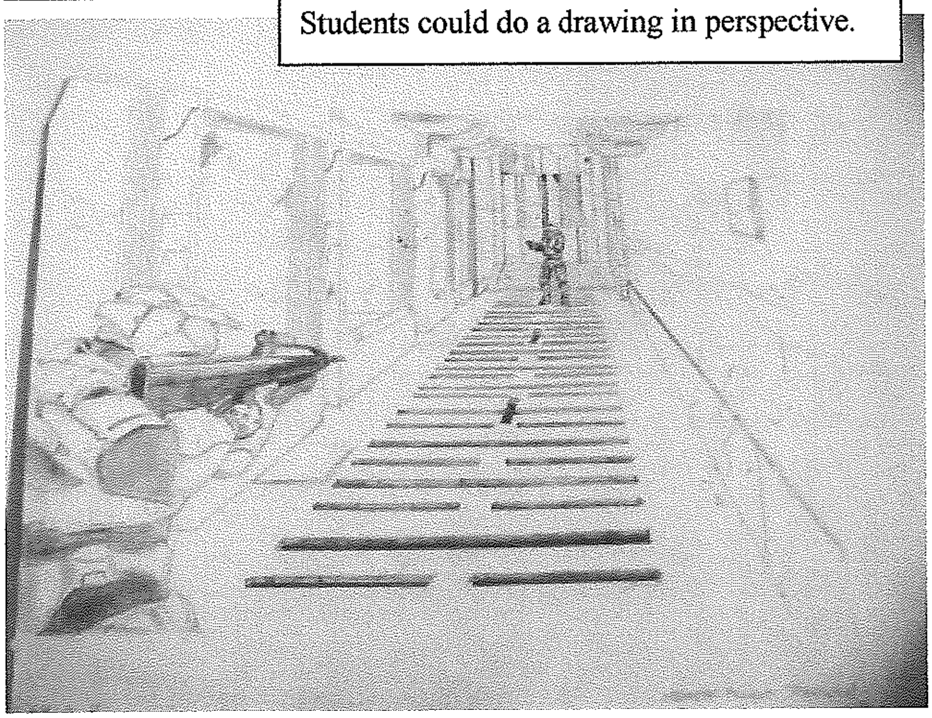
My only regret is that there was not enough time to do all that I had originally planned. I would have liked to involve the Lenart spheres at some point and given students more activities using art. Students really enjoyed using comic book art to find the vanishing point and orthogonals. I also would have liked them to all have an opportunity to draw more. I was amazed at how many of them enjoyed this part. As a high school student, I would have loathed that part. Each student was required to complete a note guide including a drawing of a tile floor using the Alberti 3-stage method and an illustration of the theorem of Desargues.

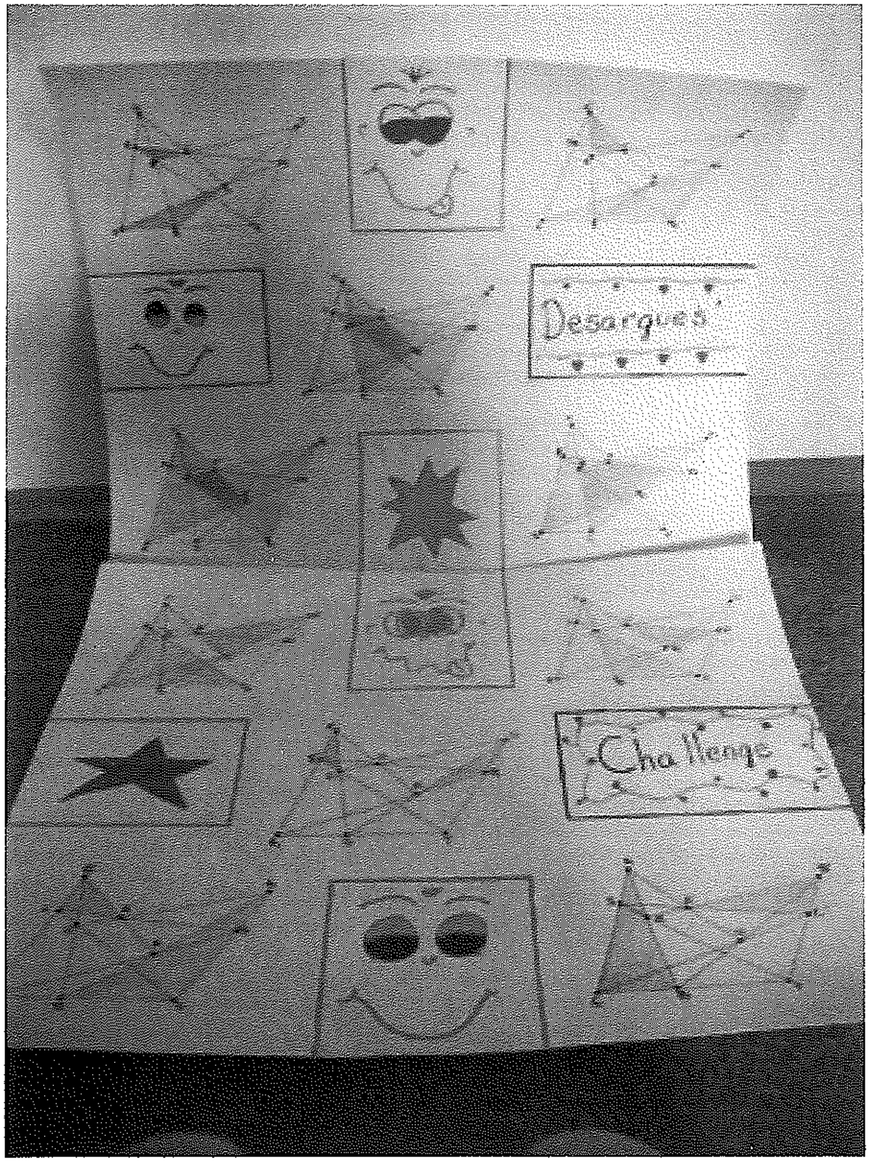
Lastly, as stated previously, my favorite part of this project was the projects done by my students. I was very lucky to have such eager learners as these amazing children. I decided to keep just a few of the good projects to use in my presentation but ended up keeping two large bags full. They continually amaze me.

Examples of Student Projects: Students had a choice of many projects or one that they created themselves. These are just a few of my favorites. The main idea was to show me that they learned something new about perspective, geometry and the Renaissance.

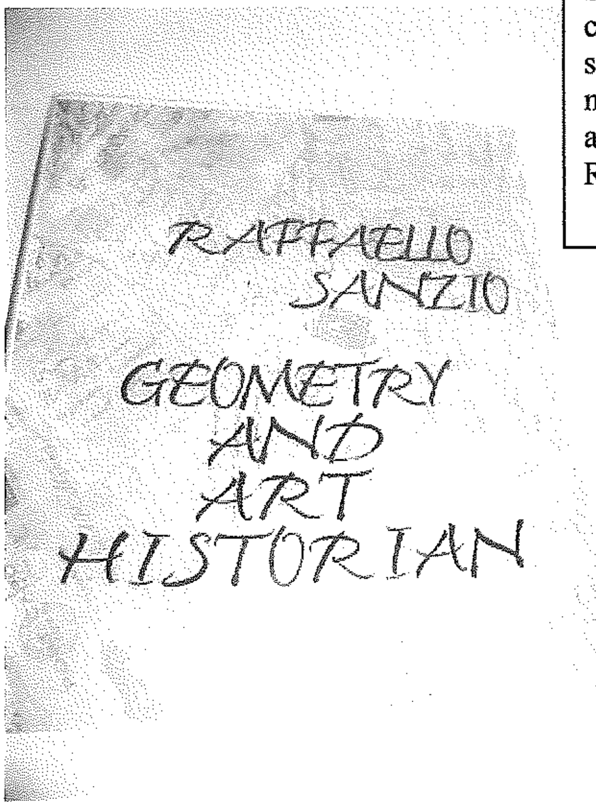


Students could do a drawing in perspective.



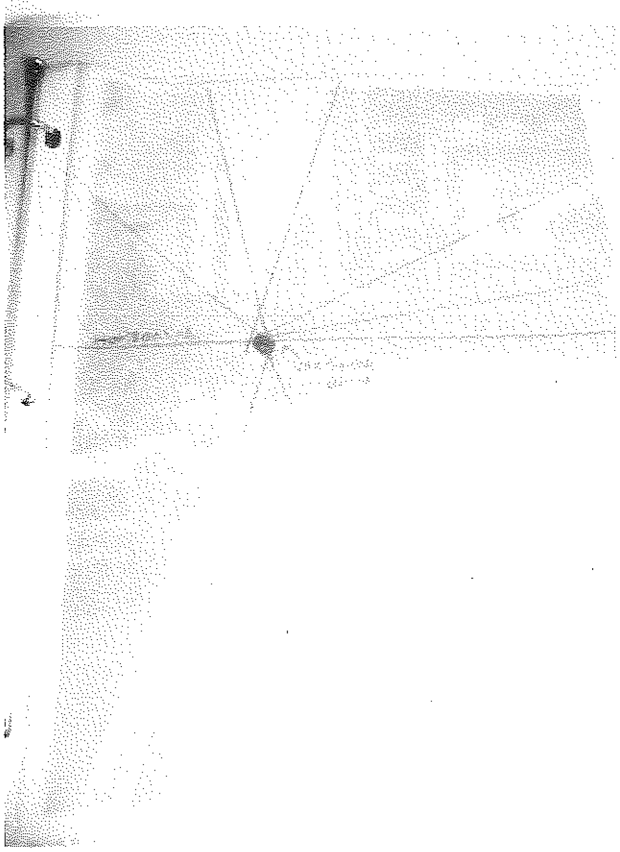


This is an example of the Desarguean configuration where all ten points could be the center of perspectivity and all other points follow.

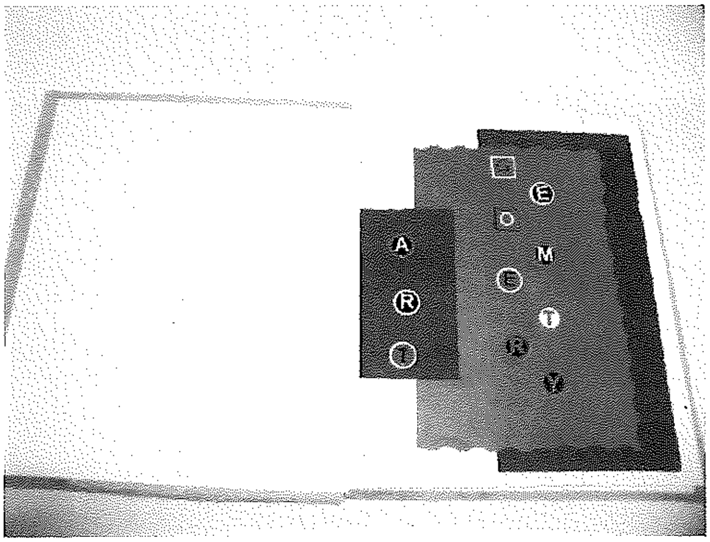


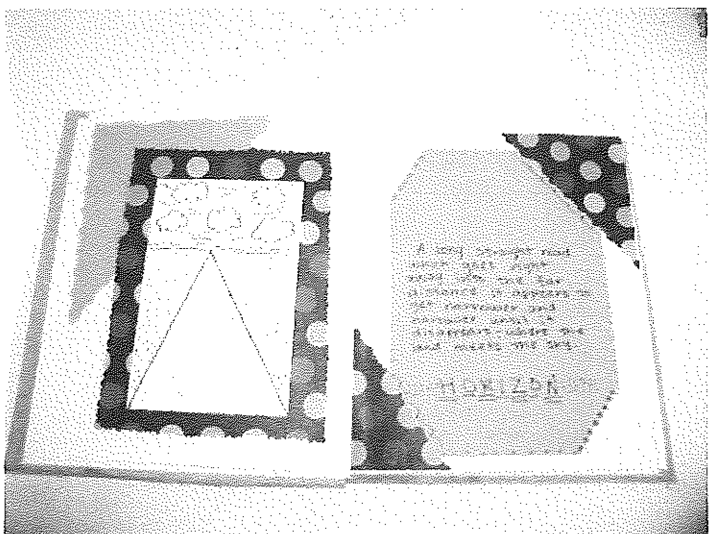
Students could choose to write a short paper on a mathematician or artist of the Renaissance.

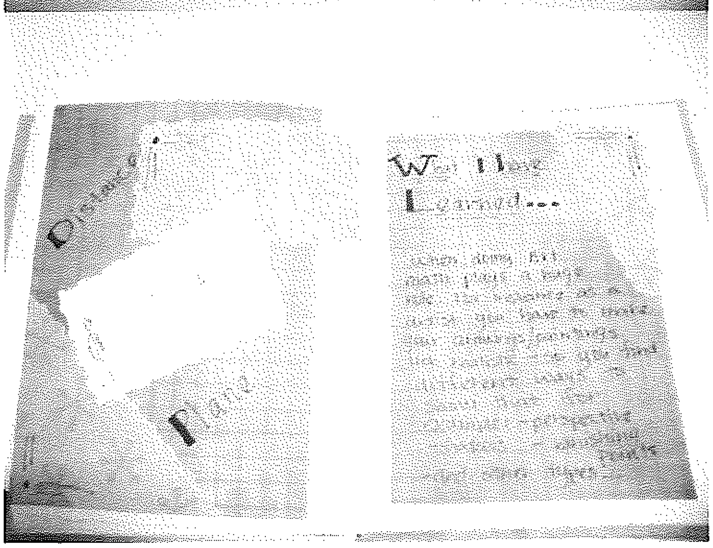
This is another example of showing perspective.



Student could take photos and on tracing paper show the vanishing point and orthogonals. Photos can be from magazines, the internet or from photos taken by students. They were required to show at least 4 examples.

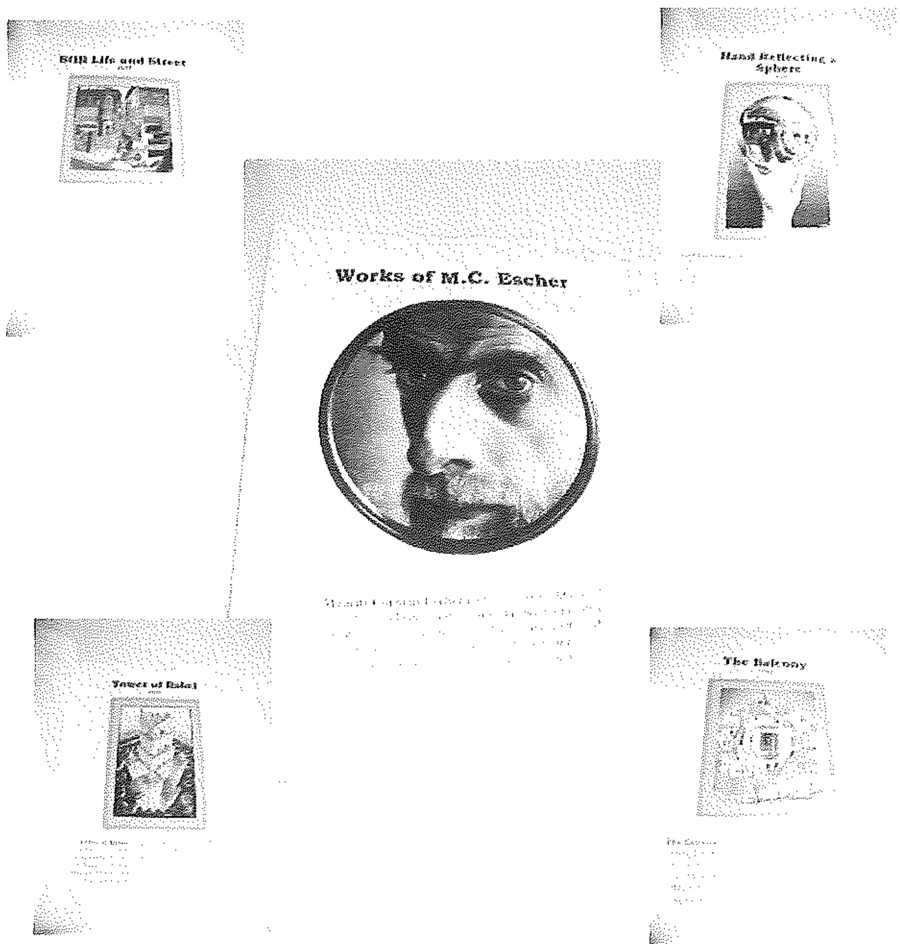






Students could create a memory book or scrapbook showing examples of what they learned.

Another example of researching the use of perspective by artists. This project looked at 11 of M. C. Escher's works.



MATH AND ART:
A PERSPECTIVE INTO PERSPECTIVE

Jason Frank D2 p13

+20

THE HISTORY OF PERSPECTIVE
THE THEOREM OF DESARGUES

BY SHELLEY LOPRESTI

WHAT IS PERSPECTIVE?

PROFESSOR PERSPECTIVE WILL SHOW YOU...IS IT ART? IS IT MATH? OR IS IT BOTH??? TAKE A JOURNEY WITH THE PROFESSOR AND FIND OUT.

The technique of representing 3D objects & Depth relationships on a 2D surface.

INTERIOR ART BY AARON LOPRESTI WITH PERMISSION FROM MARVEL COMICS. PROFESSOR PERSPECTIVE CREATED BY AARON AND SHELLEY LOPRESTI.

EGYPTIAN

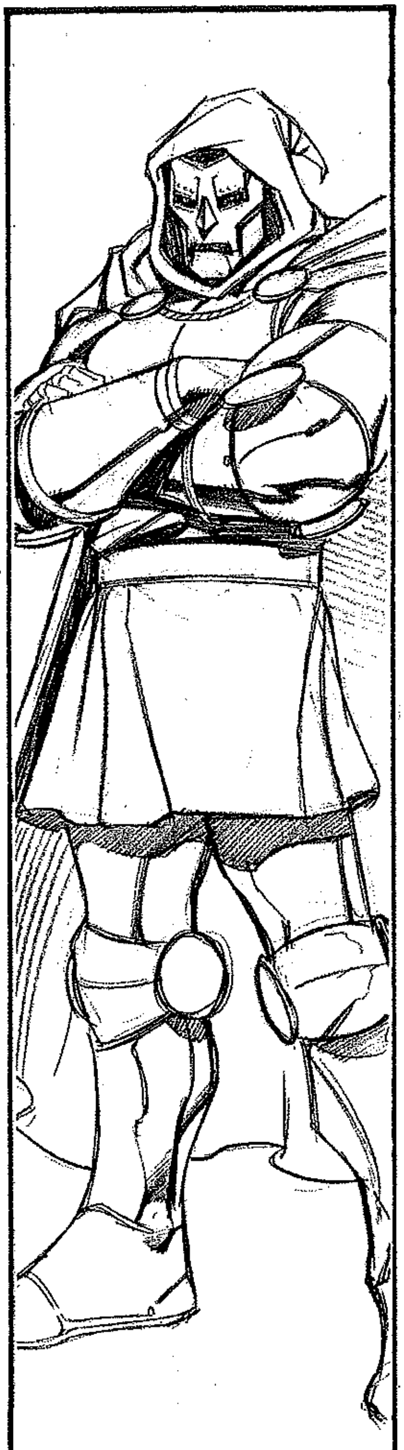
-Represented religious & social symbols
- By overlapping objects depth was suggested
- Seperate views within a figure
- Lacked realism.

GREEK AND ROMAN

- Observation
Perspective was used. (Part of figure was foreshortened to create perspective but was not complete).
-Roman theatrical scene painters developed a system based on optics to create perspective. -popular
-When Roman Empire fell, all new ideas were rejected.
-First for rich, then modest citizens.

MIDDLE AGES

- Art appears to be flat.
- Middle Ages led to the beginning of the Renaissance period which is known for the advances in art & sciences.



RENAISSANCE

- The idea to accurately depict the world as seen through a window
- Italian artists combined natural observation with simple measurement.

RAFFAELO SANZIO - he was considered the greatest painter of his time. He was also an architect.

Northern European Artists -
- Method of Judging by eye lacked a mathematical foundation.
- used convex mirrors as a compositional aid. Ex: Arnolfini Marriage.
- Artists' skills now viewed as a noble profession & not as a medieval craft
- Perspective explained as the same principal as archery. (The eye perceives visual rays from an objects that converge at the eye. This is know as a visual pyramid.)
- Credit for discovery goes to the Florentines.

Filippo Brunelleschi - Famous "peepshow" experiments that created illusion of depth based on vanishing point.
- coined creation of linear perspective.

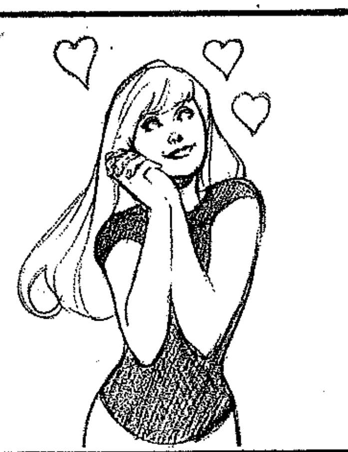
Leon Battista Alberti - Methods for drawing in perspective. - 3 stage tech.
- Alberti veil.

Leonardo Da Vinci - "Those who are enamored of practice without science are like sailors who board a ship w/out rudder & compass, never having any certainty as to whither they go."

Albrecht Durer - Believed GEOMETRY was foundation

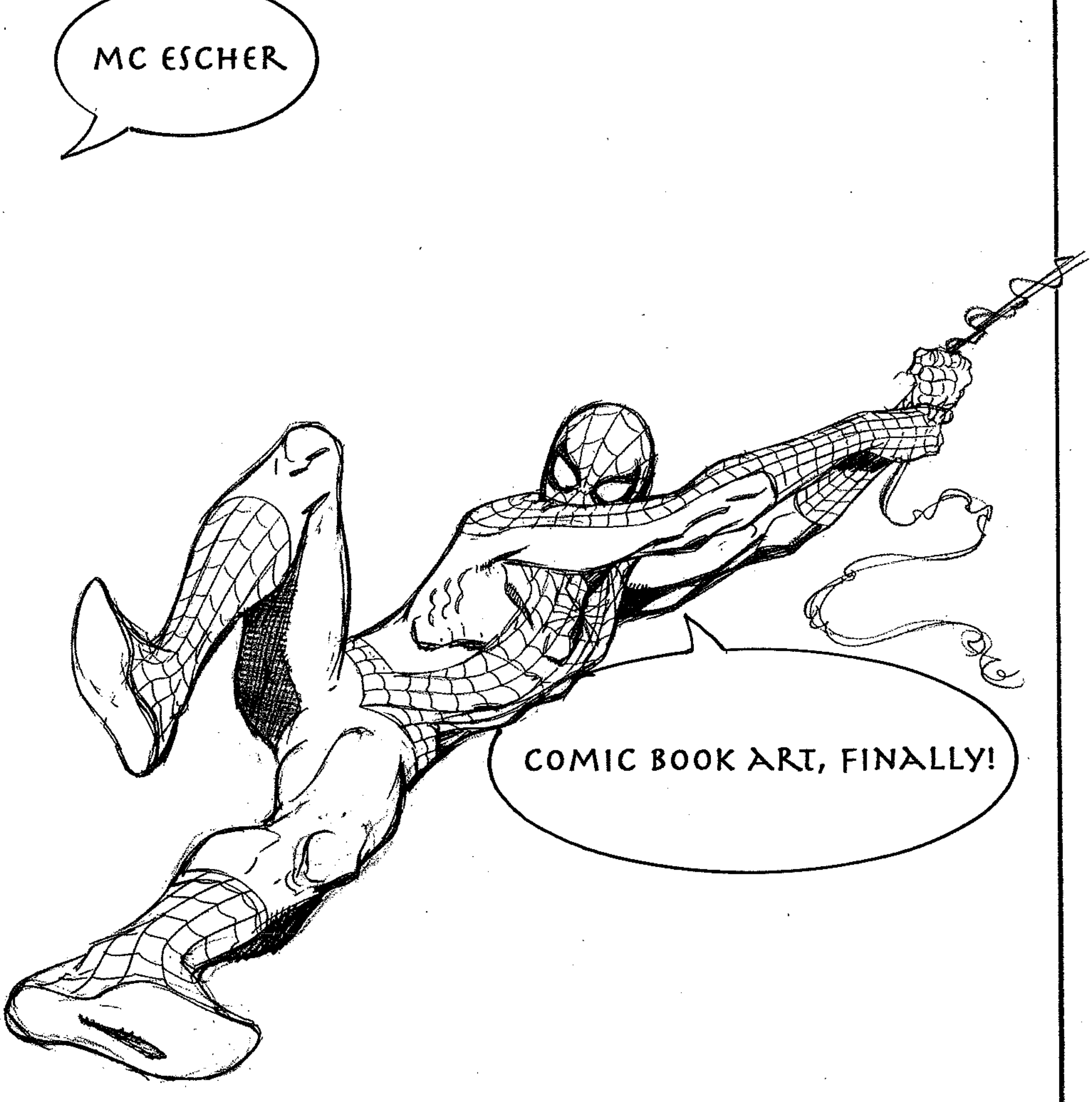
ROMANTIC

- New emphasis on the artists' imaginations
- The use of perspective to show infinity or landscapes of epic scale.
- Combo - perspective, color, & light.



MODERN ERA

Picasso, Beckman, Salvador Dali, MC Escher, used perspective to create illusions.



ROOTS OF GEOMETRY

- Roots in Babylon & Egypt

- Nile valley would flood & the harpedonopta or rope stretcher would measure land. Measured land & height of flood used to determine taxes. Could taxes have generated the need for geometry?

LOGIC

DEF: SHORT & CONCISE

AXIOMS: CREATED FROM DEFS.

THEOREMS: RELATIONSHIPS

This created rigor in a proof → Euclid's Elements

One of 1st books printed on first printing press.

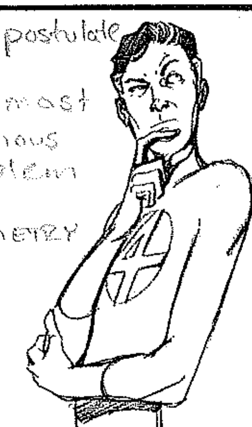
"ELEMENTS". Euclids book very influential composed of 13 books.

1st book is 23 defs. 5 postulates, 5 axioms

axioms: common notions.

5th postulate

4th most famous problem of GEOMETRY



GOLDEN AGE

- The developement of mathematics beyond calculations.

- Thales began to organize system for Geometry. - 300 BC, Euclid creates Elements.

EUCLID

COMMON NOTIONS

1.
2. a = b (a+1) = (b+1) wholes are =
3. a = b (a-1) = (b-1) remainders =
4. Things that coincide are =
5. Whole greater than part

DEFINITIONS

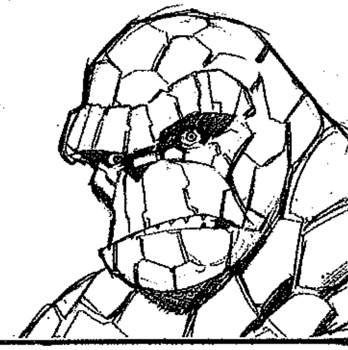
1. Point has no part
2. Line is a breadthless length
3. Ends of line are points (Line segment)
4. Straight line is straight
5. Surface

AXIOMS OR POSTULATES

1. 2 points make a line.
2. Line segment
3. To describe a circle with any center & a radius.
4. All right angles are =
5.

THEOREMS

NON-EUCLIDEAN



GAUSS

Questioning the absolute necessity of the parallel postulate.
called the parallel issue a "shamefull" part of math.
accepted that another logic system where Euclid's 5th postulate does not exist could happen.
created Hyperbolic Space. created another geometry.

RIEMANN

Student of Gauss - led us into spherical Geometry. Did not follow Euclid's postulates
based of geometry of surface, of sphere. Lines are great circles, finite, there are no parallel lines, Δ's have 3 right Δ's. Led to projective geometry.

GIRARD DESARGUES

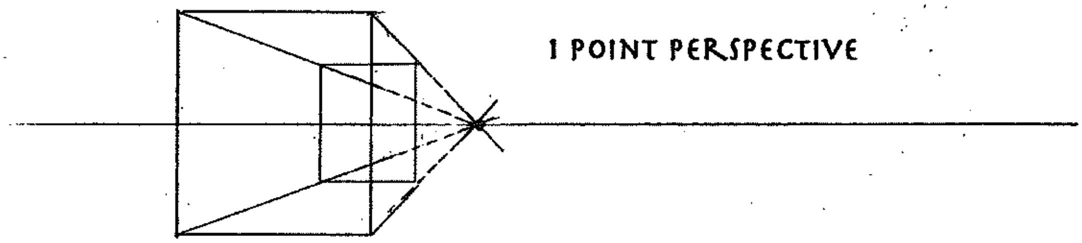
Credited w/ invention of PROJECTIVE GEOMETRY. Views considered orthodox. Born in France to a wealthy family.
Studies led to THEOREM OF DESARGUES

PROJECTIVE GEOMETRY

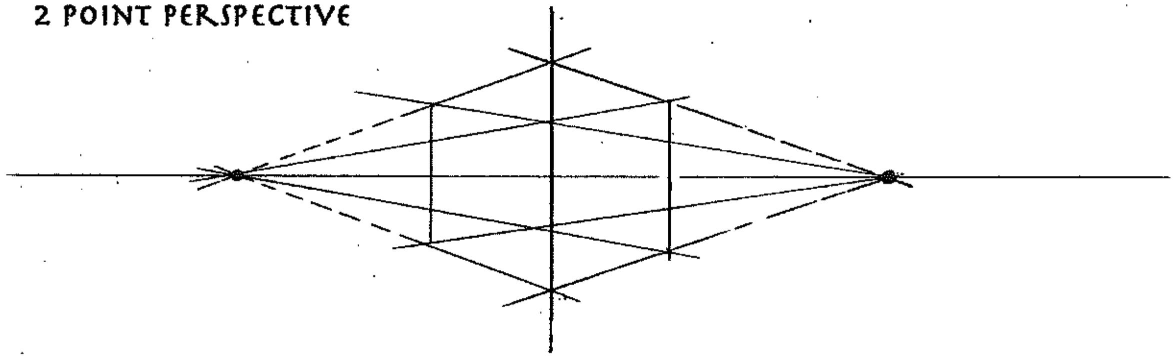
Theorem of Desargues: IF the 3 straight lines joining the corresponding verticies of 2 Δ's & all meet at a point, then the 3 intersections of pairs of corresponding sides lie on a straight line. Equivalently, if 2 Δ's are perspective from a point, they are perspective from a line.
Projective Geometry focuses on the invariant properties under projection.

VALID AND CONSISTENT

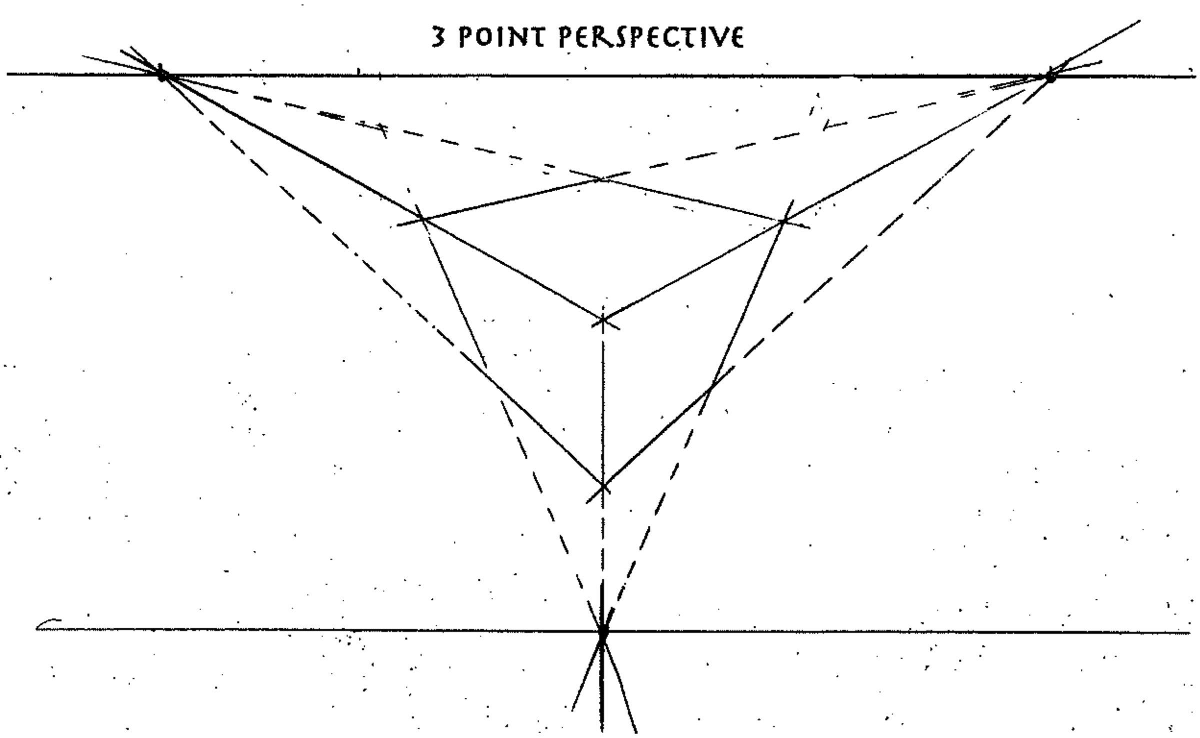
1 POINT PERSPECTIVE



2 POINT PERSPECTIVE



3 POINT PERSPECTIVE



ULTIMATE X-MEN ½ 2 ULTIMATE X-MEN ½ 3 LOPRESKI · MIKI

CAN YOU FIND THE VANISHING POINT?

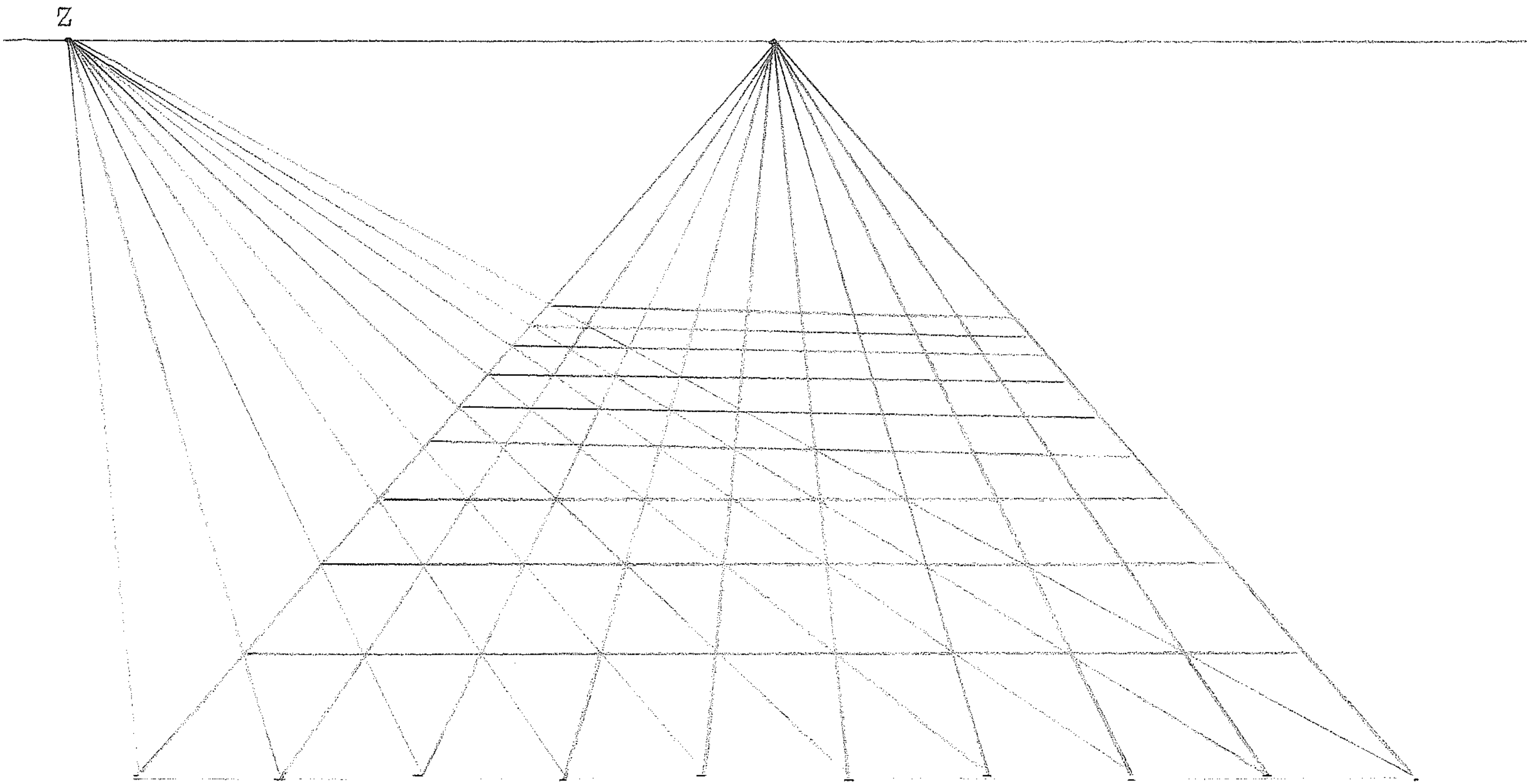
100 T

the ROGUE files 2 4


USMC
USMC
USMC
USMC

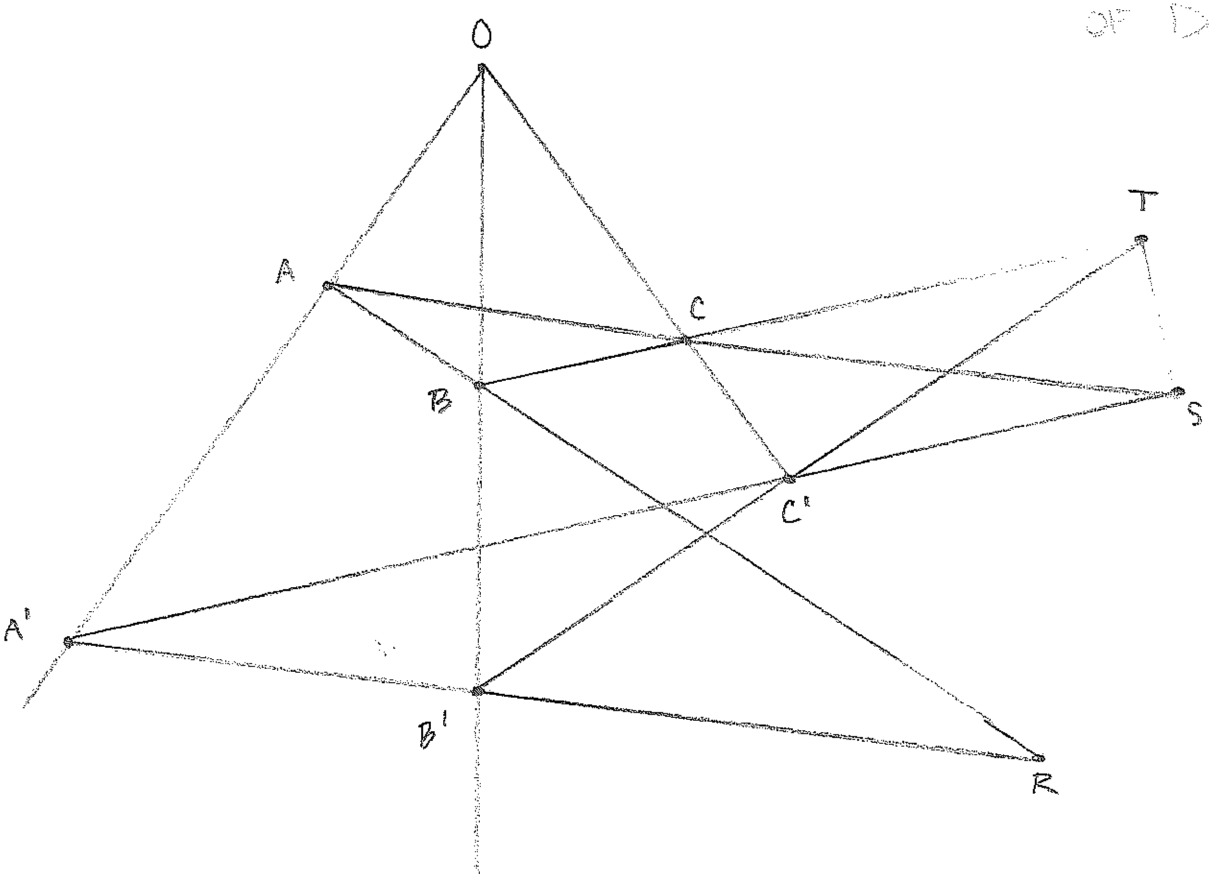
VANISHING POINT

D.2 P.3



THE THEOREM OF DESARGUES

Jason Frank
D. 2 p. 3



References

Bonola, R. (1955). *Non-euclidean geometry.* (H. S. Carslaw, Trans). New York: Dover Publications, Inc. (Original work published in 1911)

Boyer, C. (1991). *A history of mathematics.* New York: John Wiley & Sons, Inc.

Castellanos, J. (1994). *NonEuclid.* Retrieved July 14, 2003 from http://www.cs.unm.edu/~joel/NonEuclid/proof.html.

Cole, A. (1992). *Perspective.* London: Dorling Kindersley.

Cole, R. (1976). *Perspective for artists.* New York: Dover Publications, Inc.

Field, J., (August 1995). *Girard Desargues.* Retrieved July 31, 2005 from http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Desargues.html.

Geometry standards for grades 9-12. (2005). Retrieved July 27, 2005 from http://standards.nctm.or/document/chapter7/geom.htm.

Greenhood, D. (1964). *Mapping.* Chicago: The University of Chicago.

Ghyka, M. (1977). *The geometry of art and life.* New York: Dover Publications, Inc.

Heng Ser Guan, K, (2004). *Perspective in mathematics and art.* Retrieved March 12, 2004 from http://www.math.nus.edu.sg/aslaksen/projects/perspective/home2.htm.

Henle, M. (2001). *Modern geometries: Non-euclidean, projective and discrete.* Upper Saddle River: Prentice Hall.

Kaplan, R. & E. Kaplan. (2003). *The art of the infinite: The pleasures of mathematics*. New York: Oxford University Press.

Kinsey, L. & T. Moore. (2002). *Symmetry, shape, and space: An introduction to mathematics through Geometry*. Emeryville: Key College Publishing.

Lee, X. (2004). *Introduction to the real projective plane*. Retrieved August 21, 2005 from http://www.xahlee.org/projective_geometry_projective_geometry.html.

Map Projections Overview. (2000). Retrieved July 28, 2005, from http://www.colorado.edu/geography/gcraft/notes/mapproj_ftoc.html.

Meserve, B. (1983). *Fundamental concepts of geometry*. New York: Dover Publications, Inc.

Mlodinow, L. (2001). *Euclid's window*. New York: Touchstone.

O'Connor, J., & E. Robertson., (February 1996). *Non-euclidean geometry*. Retrieved July 14, 2003 from http://www-gap.dcs.st-and.ac.uk/~history/HistTopics/Non-Euclidean_geometry.html.

O'Connor, J., & E. Robertson., (February 2002). *Filippo Brunelleschi*. Retrieved July 14, 2003 from http://www-gap.dcs.stand.ac.uk/~history/Mathematicians/Brunelleschi.html.

Perspective. (2006). Wikipedia. Retrieved March 22, 2006, from http://www.wikipedia.com.

Pedoe, D. (1970). *Geometry: A comprehensive course.* New York: Dover Publications, Inc.

Pedoe, D. (1976). *Geometry and the arts.* New York: Dover Publications, Inc.

Rucker, R., (1977). *Geometry, relativity and the fourth dimension.* New York: Dover Publications, Inc.